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Short communication

Distribution of a linear function of correlated ordered variables

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ABSTRACT

In this paper, we have considered the problem of finding the distribution of a linear combination of the minimum and the maximum for a general bivariate distribution. The general results are used to obtain the required distribution in the case of bivariate normal, bivariate exponential of Arnold and Strauss, absolutely continuous bivariate exponential distribution of Block and Basu, bivariate exponential distribution of Raftery, Freund's bivariate exponential distribution and Gumbel's bivariate exponential distribution. The distributions of the minimum and maximum are obtained as special cases.

1. Introduction

Consider a random variable (X, Y) having a $BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. The distributions of $X_{(1)} = \text{Min}(X, Y)$ and $X_{(2)} = \text{Max}(X, Y)$ have been of interest in recent years and have been obtained by Cain (1994), Gupta and Gupta (2001) and Behboodian et al. (2006). The monotonicity of the failure rates of $X_{(1)}$ and $X_{(2)}$ has been studied by Gupta and Gupta (2001) and it has been shown that $X_{(1)}$ and $X_{(2)}$ have the *IFR* (increasing failure rate) property. For any bivariate distribution, general formulas for the distributions of $X_{(1)}$ and $X_{(2)}$ have been obtained by Gupta and Gupta (2001) and Gupta et al. (2006).

In this paper, we are interested in the distribution of $U = a_1X_{(1)} + a_2X_{(2)}$, where a_1 and a_2 are real constants and (X, Y) has any continuous bivariate distribution. In this context, Nagaraja (1982) obtained the distribution of U when (X, Y) has a standard bivariate normal distribution. The results of Nagaraja (1982) correct the earlier derivation of Gupta and Pillai (1965). Recently Genc (2006) addressed the same problem and obtained the distribution of U in the case of $BVN(0, 0, \sigma_1^2, \sigma_2^2, \rho)$. Instead of confining our attention to only bivariate normal distribution, we obtain the distribution of U for any continuous bivariate distribution. The result for bivariate normal distribution is derived as a special case. Some more examples are provided in the case of Arnold and Strauss (1988) bivariate exponential distribution, absolutely continuous bivariate distribution of Block and Basu (1974), bivariate exponential distribution of Raftery (1984), Freund's bivariate exponential distribution and Gumbel's bivariate exponential distribution.

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2. Distribution of a linear combination

Here we obtain the cumulative distribution function (*cdf*) and the probability density function (*pdf*) of $U = a_1X_{(1)} + a_2X_{(2)}$ for any absolutely continuous random variable (X, Y) with joint *pdf* $f(x, y)$.

Theorem 2.1. *The cdf and pdf of $U = a_1X_{(1)} + a_2X_{(2)}$, where a_1 and a_2 are real constants, $a_1 + a_2 \neq 0$, are given by*

$$F_U(t) = \int_{u=-\infty}^{t}\int_{v=-\infty}^{0} f_{X,Y}\left(\frac{u+a_2v}{a_1+a_2}, \frac{u-a_1v}{a_1+a_2}\right)\frac{1}{|a_1+a_2|}\,\mathrm{d}v\,\mathrm{d}u + \int_{u=-\infty}^{t}\int_{v=0}^{\infty} f_{X,Y}\left(\frac{u+a_1v}{a_1+a_2}, \frac{u-a_2v}{a_1+a_2}\right)\frac{1}{|a_1+a_2|}\,\mathrm{d}v\,\mathrm{d}u \tag{2.1}$$

and

$$f_U(t) = \int_{v=-\infty}^{0} f_{X,Y}\left(\frac{t+a_2v}{a_1+a_2}, \frac{t-a_1v}{a_1+a_2}\right)\frac{1}{|a_1+a_2|}\,\mathrm{d}v + \int_{v=0}^{\infty} f_{X,Y}\left(\frac{t+a_1v}{a_1+a_2}, \frac{t-a_2v}{a_1+a_2}\right)\frac{1}{|a_1+a_2|}\,\mathrm{d}v. \tag{2.2}$$

Proof.

$$P(a_1X_{(1)} + a_2X_{(2)} \leqslant t) = P(a_1X + a_2Y \leqslant t, X < Y) + P(a_2X + a_1Y \leqslant t, X > Y).$$

Making the substitutions $a_1X + a_2Y = U_1$ and $X - Y = V$, it can be verified that

$$P(a_1X + a_2Y \leqslant t, X < Y) = P(U_1 \leqslant t, V < 0)$$
$$= \int_{u=-\infty}^{t}\int_{v=-\infty}^{0} f_{X,Y}\left(\frac{u+a_2v}{a_1+a_2}, \frac{u-a_1v}{a_1+a_2}\right)\frac{1}{|a_1+a_2|}\,\mathrm{d}v\,\mathrm{d}u. \tag{2.3}$$

Similarly, let $a_2X + a_1Y = U_2$ and $X - Y = V$. Then we have

$$P(a_2X + a_1Y \leqslant t, X > Y) = P(U_2 \leqslant t, V > 0) = \int_{u=-\infty}^{t}\int_{v=0}^{\infty} f_{X,Y}\left(\frac{u+a_1v}{a_1+a_2}, \frac{u-a_2v}{a_1+a_2}\right)\frac{1}{|a_1+a_2|}\,\mathrm{d}v\,\mathrm{d}u. \tag{2.4}$$

Combining (2.3) and (2.4), we get (2.1). From (2.1), one can readily obtain (2.2). This establishes the results. □

Special cases:

1. $a_1 = 1, a_2 = 0, T_1 = \mathrm{Min}(X, Y)$.
   Using (2.2) we have

$$f_{T_1}(t) = \int_t^{\infty} f_{X,Y}(t, \psi)\,\mathrm{d}\psi + \int_t^{\infty} f_{X,Y}(\psi, t)\,\mathrm{d}\psi$$
$$= f_X(t)P(Y > t|X = t) + f_Y(t)P(X > t|Y = t), \tag{2.5}$$

   where $f_X(t)$ and $f_Y(t)$ are the marginal *pdf*'s of X and Y, respectively. This conforms with Gupta et al. (2006).
2. $a_1 = 0, a_2 = 1, T_2 = \mathrm{Max}(X, Y)$.
   Proceeding as before, we see that

$$f_{T_2}(t) = \int_{-\infty}^{t} f_{X,Y}(\psi, t)\,\mathrm{d}\psi + \int_{-\infty}^{t} f_{X,Y}(t, \psi)\,\mathrm{d}\psi$$
$$= f_X(t)P(Y < t|X = t) + f_Y(t)P(X < t|Y = t), \tag{2.6}$$

   same as in Gupta et al. (2006).

In the following, we present the result when $a_1 + a_2 = 0$

Theorem 2.2. *The pdf of $U = a_1X_{(1)} + a_2X_{(2)}$, where a_1 and a_2 are real constants, $a_1 + a_2 = 0$, is given by*

$$f_U(t) = \frac{1}{|a_2|}\int_{-\infty}^{\infty} [f_{X,Y}(x, x + t/a_2) + f_{X,Y}(x, x - t/a_2)]\,\mathrm{d}x. \tag{2.7}$$

Proof. In this case $U = a_1X_{(1)} + a_2X_{(2)} = a_2|Y - X|$.