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An alternative form of the Watson efficiency

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ABSTRACT

Watson [1951. Serial correlation in regression analysis. Ph.D. Thesis, Department of Experimental Statistics, North Carolina State College, Raleigh] introduced a relative efficiency, which is often called the Watson efficiency in literatures, to measure the inefficiency of the least squares in linear regression models. The Watson efficiency is defined by determinant, but we shall show by two examples that such a criterion does not always work well in some cases. In this paper, an alternative form based on Euclidean norm of the Watson efficiency is proposed and some examples are given to illustrate superiority of the new relative efficiency.

1. Introduction

Consider the following linear regression model:

$$y = X\beta + \varepsilon, \tag{1.1}$$

where $y \in R^n$ is the vector of n observations on the study variable, $X \in R^{n\times p}$ is the known design matrix with full column rank p, $\beta \in R^p$ is the unknown vector of regression coefficients and $\varepsilon \in R^n$ is the error vector with mean vector zero and covariance matrix $\sigma^2\Sigma$, here Σ is assumed to be positive definite, denoted as $\Sigma > 0$.

It is known that the best linear unbiased estimator (BLUE) of β is given by

$$\widetilde{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y \tag{1.2}$$

and the ordinary least squares estimator (OLSE) of β is given by

$$\widehat{\beta} = (X'X)^{-1}X'y. \tag{1.3}$$

Since both the BLUE and OLSE are unbiased estimators of β, we naturally want to know which one is better. Fisher (1922) introduced the concept of efficiency based on variance to measure the relative merits of a few estimators. That is, an estimator is viewed as more efficient if it has smaller variance. We can easily compute from (1.2) and (1.3) that

$$\Delta = cov(\widehat{\beta}) - cov(\widetilde{\beta}) = \sigma^2[(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1} - (X'X)^{-1}X']\Sigma[(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1} - (X'X)^{-1}X']', \tag{1.4}$$

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which is clearly nonnegative definite. Many authors have discussed the conditions under which $\Delta = 0$, see, e.g., Rao (1968), Watson (1967, 1972), Krämer (1980), Puntanen (1986) and Baksalary (1988).

In order to compare the OLSE with BLUE, Watson (1951) introduced the relative efficiency ϕ defined as follows:

$$\phi = \frac{|cov(\widetilde{\beta})|}{|cov(\widehat{\beta})|} = \frac{|X'X|^2}{|X'\Sigma X| \cdot |X'\Sigma^{-1}X|}, \tag{1.5}$$

where $|\cdot|$ indicates the determinant of the matrix concerned. From (1.4), it is easy to get that the Watson efficiency $\phi \leqslant 1$. A lower bound for the Watson efficiency (1.5) is given by the following inequality:

$$\phi \geqslant \prod_{i=1}^{m} \frac{4\lambda_i \lambda_{n-i+1}}{(\lambda_i + \lambda_{n-i+1})^2}, \tag{1.6}$$

where $m = \min\{p, n-p\}$ and $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ are the eigenvalues of Σ. Inequality (1.6) was originally conjectured by Durbin, see Watson (1951, 1955). Then, it was almost simultaneously proved for the case $m \geqslant 2$ by Bloomfield and Watson (1975) and Knott (1975), respectively.

Furthermore, Chu et al. (2004, 2005, 2007) considered the general partitioned linear regression model

$$y = X_1\beta_1 + X_2\beta_2 + e, \tag{1.7}$$

and introduced the following two subset Watson efficiencies:

$$\phi_1(\widehat{\beta}_1|\mathscr{M}_{12}) = \frac{|cov(\widetilde{\beta}_1|\mathscr{M}_{12})|}{|cov(\widehat{\beta}_1|\mathscr{M}_{12})|} \quad \text{and} \quad \phi_2(\widehat{\beta}_2|\mathscr{M}_{12}) = \frac{|cov(\widetilde{\beta}_2|\mathscr{M}_{12})|}{|cov(\widehat{\beta}_2|\mathscr{M}_{12})|}, \tag{1.8}$$

where $\mathscr{M}_{12} = (X_1 \vdots X_2)$.

Both the Watson efficiency (1.5) and the subset Watson efficiencies (1.8) are defined by determinant of the concerned matrix. However, as we shall show in Examples 1 and 2 in Section 3, such criterions are not always so satisfactory. Various other relative efficiencies have also been considered in literatures, see, e.g., Khatri and Rao (1981, 1982), Styan (1983), Rao (1985), Yang (1988, 1990, 1996) and Wang and Yang (1989). In this paper, we shall consider an alternative relative efficiency based on Euclidean norm.

The rest of this paper is organized as follows. The definitions of our proposed relative efficiencies with their lower and upper bounds are given in Section 2. Then, two numerical examples are given to illustrate our theoretical findings in Section 3 and proofs of the main theorems are provided in Appendix.

2. The new relative efficiency

In this section, we shall firstly introduce a new relative efficiency δ, which is defined as follows:

$$\delta = \frac{\|\mathrm{cov}(\widetilde{\beta})\|_F}{\|\mathrm{cov}(\widehat{\beta})\|_F} = \frac{\|(X'\Sigma^{-1}X)^{-1}\|_F}{\|(X'X)^{-1}X'\Sigma X(X'X)^{-1}\|_F}, \tag{2.1}$$

where $\|\cdot\|_F$ denotes Euclidean norm (or Frobenius norm) of the matrix concerned. To make the lower bound of δ be invariant for the choice of design matrix X, we shall also consider the following relative efficiency:

$$\rho = \frac{\|\mathrm{cov}(X\widetilde{\beta})\|_F}{\|\mathrm{cov}(X\widehat{\beta})\|_F} = \frac{\|X(X'\Sigma^{-1}X)^{-1}X'\|_F}{\|X(X'X)^{-1}X'\Sigma X(X'X)^{-1}X'\|_F}. \tag{2.2}$$

It is easy to prove that $\delta \leqslant 1, \rho \leqslant 1$ by (1.4).

Theorem 1. *Let Σ be an $n \times n$ symmetry positive definite matrix, X be an $n \times p$ matrix such that $X'X = I_p$, and $\lambda_1 \geqslant \cdots \geqslant \lambda_n > 0$ be the ordered eigenvalues of Σ. If $p < n$, then we have*

$$\frac{\|X'\Sigma X\|_F}{\|(X'\Sigma^{-1}X)^{-1}\|_F} \leqslant \frac{1}{2p}\left(\sqrt{\frac{\lambda_1\lambda_{n-p+1}}{\lambda_n\lambda_p}} + \sqrt{\frac{\lambda_n\lambda_p}{\lambda_1\lambda_{n-p+1}}}\right)\sum_{i=1}^{p}\frac{\lambda_i}{\lambda_{n-p+i}}$$

and the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$.

Proof. See Appendix.