Journal of Statistical Planning and Inference 139 (2009) 1484 – 1492

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Bayesian modeling using a class of bimodal skew-elliptical distributions

David Elal-Olivero[a,1], Héctor W. Gómez[b,2], Fernando A. Quintana[c,*,3]

[a] Departamento de Matemáticas, Facultad de Ingeniería, Universidad de Atacama, Chile
[b] Departamento de Estadística, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Concepción, Chile
[c] Departamento de Estadística, Facultad de Matemática, Pontificia Universidad Católica de Chile, Santiago, Chile

ARTICLE INFO

Article history:
Received 13 December 2006
Received in revised form
26 September 2007
Accepted 23 July 2008
Available online 20 August 2008

Keywords:
Bimodality
Density estimation
Linear regression
Skew-normal distribution
Stochastic representation

ABSTRACT

We consider Bayesian inference using an extension of the family of skew-elliptical distributions studied by Azzalini [1985. A class of distributions which includes the normal ones. Scand. J. Statist. Theory and Applications 12 (2), 171–178]. This new class is referred to as bimodal skew-elliptical (BSE) distributions. The elements of the BSE class can take quite different forms. In particular, they can adopt both uni- and bimodal shapes. The bimodal case behaves similarly to mixtures of two symmetric distributions and we compare inference under the BSE family with the specific case of mixtures of two normal distributions. We study the main properties of the general class and illustrate its applications to two problems involving density estimation and linear regression.

© 2008 Elsevier B.V. All rights reserved.

1. Introduction

Azzalini and Capitanio (2003) and Azzalini (2005) proved that if g is a probability density function (pdf) that is symmetric about zero and if H is a cumulative distribution function (cdf) such that its density h is also symmetric about zero, then for any odd function $w(x)$

$$f_X(x) = 2g(x)H(w(x)), \quad -\infty < x < \infty, \tag{1}$$

is a pdf on $\mathbb{R}$. This result has turned out to be quite useful in the process of constructing skewed distributions from symmetric ones. We denote a random variable X with pdf (1) as $X \sim S(g,h,w)$. In particular, if $g=\phi$, and $H=\Phi$ represent the standard normal pdf and cdf, respectively, then taking $w(x)=\lambda x$ for $\lambda \in \mathbb{R}$ we obtain the skew-normal (SN) distribution. This has been extensively studied in Azzalini (1985, 1986), Henze (1986), Pewsey (2000) and many others. The special case where $H'=g$ and g is the Laplace, logistic or uniform distribution, has been considered by Gupta et al. (2002). Related constructions have been developed in Balakrishnan and Ambagaspitiya (1994) and in Arnold and Beaver (2000a, b). Nadarajah and Kotz (2003) consider fixing $g=\phi$ and H given by one of the normal, Student-t, Cauchy, Laplace, logistic or uniform cdfs. Gómez et al. (2007) consider the family of distributions generated by fixing $H=\Phi$ and letting g be any symmetric pdf. More generally, there has been a number of works exploring bimodality arising from skew-distributions. Azzalini and Capitanio (2003), Arnold et al. (2002), Ma and Genton (2004), Arellano-Valle et al. (2005, 2006).

* Corresponding author.
E-mail addresses: delal@mat.uda.cl (D. Elal-Olivero), hgomez@udec.cl (H.W. Gómez), quintana@mat.puc.cl (F.A. Quintana).

[1] Partially supported by Grant DIUDA 221140.
[2] Partially supported by Grant FONDECYT 1060727.
[3] Partially supported by Grant FONDECYT 1060729.

0378-3758/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.jspi.2008.07.016

From the Bayesian standpoint, inference for skew-elliptical models has been considered in Fernández and Steel (1998) and Sahu et al. (2003). More recently, Ferreira and Steel (2006) described a general form to introduce skewness in symmetric models.

Our aim is to introduce a new family of distributions that is flexible enough to support both, uni- and bimodal shapes. Many datasets arising in practice can be adequately modeled this way and so our proposal plays a unifying role in this context.

To achieve our purpose, we resort to (1), adopting a specific form for $w(x)$ that yields the desired result. We will study properties of the proposed family of distributions as well as applications to some standard problems. In the theoretical discussion, the following stochastic representation of (1) turns out to be quite useful: let $X \sim g$ and $Y \sim h$ be independent random variables and define

$$Z = \begin{cases} X & \text{if } Y < w(X), \\ -X & \text{if } Y \geqslant w(X). \end{cases} \tag{2}$$

Then Z has distribution given by (1). This shows that these skew-distributions arise from a hidden truncation mechanism (Ferreira and Steel, 2006).

The rest of this paper is organized as follows. Section 2 presents the new family, and develops its main property. In particular, we show how uni- and bimodal shapes are obtained. Section 3 discusses some practical issues in the general use of skew-bimodal distributions in Bayesian inference. Section 4 illustrates the use of SN with bimodal shape distributions for density and linear regression problems. Special attention is given to comparing inference under the new family with mixtures of two normal distributions. A final discussion is presented in Section 5.

2. Adding bimodality and skewness to symmetric distributions

The result stated next is useful in the process of introducing skewness *and* bimodality in families of symmetric distributions.

Proposition 1. *Let f_0 be a unimodal pdf, H a cdf having pdf h, and assume both f_0 and h are symmetric about zero. If we further assume $\kappa = \int_{-\infty}^{\infty} x^2 f_0(x)\,dx < \infty$, then*

$$f(x) = 2\left(\frac{1+\alpha x^2}{1+\alpha\kappa}\right) f_0(x) H(w(x)), \quad -\infty < x < \infty, \tag{3}$$

is a pdf for any odd function $w(\cdot)$ and any $\alpha \geqslant 0$.

To prove Proposition 1 consider, for a given $\alpha \geqslant 0$,

$$g(x) = \left(\frac{1+\alpha x^2}{1+\alpha\kappa}\right) f_0(x), \tag{4}$$

which is a symmetric pdf. The result then follows immediately from (1).

An immediate consequence of Proposition 1 is the stochastic representation of the new family by means of hidden truncation: if $X \sim ((1+\alpha x^2)/(1+\alpha\kappa)) f_0(x)$ is independent of $Y \sim h$ then Z defined in (2) has pdf given by (3).

Because f_0 is symmetric and unimodal, the density $g(x)$ defined in (4) is symmetric and may adopt both, uni- and bimodal shapes. Besides, it is the $H(w(x))$ factor that breaks the symmetry. For later reference we denote a random variable X with pdf $g(x)$ given in (4) by $X \sim B(f_0, \alpha)$ and a random variable Y with pdf (3) as $Y \sim SB(f_0, h, w, \alpha)$. We refer to the corresponding families of distributions that follow by letting f_0 be any symmetric unimodal pdf as *symmetric bimodal* and *skew-elliptical bimodal*, respectively. We note that the $B(f_0, \alpha)$ family contains all the symmetric unimodal distributions (taking $\alpha = 0$), and similarly, the $SB(f_0, h, w, 0)$ sub-family agrees with the class of all unimodal skew-elliptical distributions.

The above construction can be seen to generalize some well-known classes of skewed distribution. For instance, taking $f_0(x) = h(x) = \phi(x)$ and $w(x) = \lambda x$ we extend the SN model studied in Azzalini (1985). When $w(x) = \lambda x + \gamma x^3$ we obtain an extension of the model presented in Ma and Genton (2004). More generally, if $f_0 = \phi$ we generalize the family of distributions in Nadarajah and Kotz (2003) and when $h = \phi$, a generalization of Gómez et al. (2007) follows. In all these cases, the original family is recovered by choosing $\alpha = 0$.

Fig. 1 shows the effect of the α parameter on the symmetric and skewed cases, using $f_0(x) = h(x) = \phi(x)$, and a skewing factor defined in terms of $w(x) = x/2$, i.e., corresponding to the SN density with parameter $\lambda = \frac{1}{2}$. Because in general $H(w(x)) = 1 - H(-w(x))$, the α parameter can be seen as controlling the density value at the highest mode, i.e., the one concentrating most of the mass. Taking $w(x) = -x/2$ we obtain a reflection of Fig. 1 with respect to $x = 0$.

As is the case of several other families of skewed distributions, such as Azzalini's SN, the bimodal class satisfies a *perturbation invariance* property, which we state next.

Proposition 2. *If $X \sim SB(f_0, h, w, \alpha)$ then $|X| \sim HalfB(f_0, \alpha)$ for any symmetric pdf h and function w.*

Download English Version:

https://daneshyari.com/en/article/1149687

Download Persian Version:

https://daneshyari.com/article/1149687

Daneshyari.com