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Resampling schemes with low resampling intensity and their applications in testing hypotheses

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ABSTRACT

The paper explores statistical features of different resampling schemes under *low resampling intensity*. The original sample is considered in a very general framework of triangular arrays, without independence or equally distributed assumptions, although improvements under such conditions are also provided. We show that low resampling schemes have very interesting and flexible properties, providing new insights into the performance of widely used resampling methods, including subsampling, two-sample unbalanced permutation statistics or wild bootstrap. It is shown that, under regularity assumptions, resampling tests with critical values derived by the appertaining low resampling procedures are asymptotically valid and there is no loss of power compared with the power function of an ideal (but unfeasible) parametric family of tests. Moreover we show that in several contexts, including regression models, they may act as a filter for the normal part of a limit distribution, turning down the influence of outliers.

1. Introduction

In the present paper we introduce and study the concept of *low resampling intensity* for exchangeable resampling schemes. It is well known from the literature that sometimes Efron's ordinary bootstrap with resampling size $m(n)$ equal to the sample size $k(n)$ fails, while if $m(n)=o(k(n))$ it is consistent provided $\min(k(n),m(n))\to\infty$ as $n\to\infty$, see Athreya (1987) or Arcones and Giné (1989, 1991). In fact Mammen (1992a, b) showed (in an i.i.d. triangular-array setup) that, for linear statistics, consistency of the bootstrap with $m(n)=k(n)$ is equivalent to asymptotic normality. See also Cuesta-Albertos and Matrán (1998) and del Barrio et al. (1999) in relation with the general behavior of the bootstrap mean in this setup and Bickel et al. (1997) for a list of examples and further references regarding strategies to achieve bootstrap success.

The bootstrap is part of more general resampling procedures given by exchangeable weights that, as it has progressively been made apparent, include well-known statistical methods as well as some new others suggested by the framework. The behavior of the weighted bootstrap mean has been considered for instance in Mason and Newton (1992), Praestgaard and Wellner (1993), Arenal-Gutiérrez and Matrán (1996), del Barrio and Matrán (2000), Janssen and Pauls (2003) or Janssen (2005) (see the survey paper by Csörgö and Rosalsky, 2003 for other references). This general approach also covers two-sample permutation statistics, the wild bootstrap, or even subsampling (see e.g., Politis et al., 1999). Moreover, new proposals of resampling schemes designed for specific problems continuously appear in the literature. For example, Bose and Chatterjee (2005) consider generalized bootstrap

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based on resampling schemes relating the weights to estimating equations similar to those in M-estimation problems. Janssen et al. (2002, 2005) consider special bootstrap versions for quantile processes or in goodness-of-fit testing.

As we will show in this paper the nice properties of low resampling bootstrap are shared by the general low resampling schemes that we introduce here (see Definition 3). Our setup will lead to new results for

- the bootstrap with small resampling intensity,
- two-sample permutation statistics when the sample sizes n_1 and n_2, $n_1+n_2=k(n)$ are strongly unbalanced with $n_1/n_2 \to 0$, assuming $\min(n_1,n_2) \to \infty$,
- the wild bootstrap when the distributions of the external resampling variables depend on n, or
- subsampling.

The work is organized in six sections and one appendix. After some motivations and notation in this Introduction, in Section 2 we formalize the concept of low-intensity resampling and give some examples covered by this framework. In Theorem 8 we give a very general result showing different situations that arise, under low-intensity resampling, for general triangular arrays of random variables. In some sense low resampling removes the extremes (outliers) of a sample and filters the normal part. Section 3 is devoted to the row-wise independent case, and Section 4 exploits our results to give an application concerning robustness of resampling methods under low resampling. This filtering property was unknown even for some of the most common resampling methods. For instance subsampling is widely regarded to be "universally consistent'', but our results show that this is not exactly the case (see the comments following Lemma 18; see also the analysis in del Barrio et al., 2002 regarding the stability of Efron's bootstrap). We include in our Section 4 a small simulation study showing that low-intensity resampling methods do not only provide valid and robust testing procedures, but also that they can be more accurate than other non-resamplimg competitors. Then, in Section 5 we explore the application of the filtering property of low-intensity wild bootstrap in robust regression. Section 6 analyzes the unbalanced two-sample permutation tests providing a more general result than that in Janssen (1997). In the Appendix we include the proof of a technical lemma necessary for our proof of the main Theorem 8.

Throughout, we will use the notation of Janssen and Pauls (2003) and Janssen (2005). Consider a triangular array

$$X_{n,i} : (\Omega, \mathscr{A}, P) \longrightarrow \mathbb{R}, \quad 1 \leqslant i \leqslant k(n), \quad n \in \mathbb{N} \tag{1}$$

of real random variables with $k(n) \to \infty$ as $n \to \infty$. At this stage the array may be arbitrary, no independence or identical distribution is assumed so far. Thus the general results of Section 2 apply to resampling procedures for partial sums of dependent variables. Introduce

$$\vec{X}_n := (X_{n,1}, \ldots, X_{n,k(n)}), \quad \bar{X}_n := \frac{1}{k(n)} \sum_{i=1}^{k(n)} X_{n,i}. \tag{2}$$

Our paper is concerned with the conditional and unconditional correctness of (low) resampling statistics. The conditional correctness is needed for resampling tests which rely on data dependent critical values given by conditional resampling distributions. To explain this recall from Janssen and Pauls (2003, Lemma 1), that the conditional consistency of resampling critical values c_n^* is required for valid resampling tests, see also (7)–(12).

In order to motivate the results let us consider the following introductory example which is recalled from Janssen and Pauls (2003). A refinement can be achieved by studentized versions (see Janssen, 2005). For simplicity we will here restrict ourselves to the simplest case (without denominator) since we do not like to overload the paper. Further applications for low resampling statistics can be found in del Barrio et al. (2007).

Example 1 (*Continued in Section 4*). Our observations are arbitrary real random variables

$$X_{n,i} = X'_{n,i} + b_{n,i}, \quad 1 \leqslant i \leqslant k(n) \tag{3}$$

with means $E(X_{n,i}) = b_{n,i}$. The null hypothesis to be tested is

$$\mathrm{H}_0 : b_{n,i} = 0 \quad \text{for all } 1 \leqslant i \leqslant k(n) \tag{4}$$

against $\sum_{i=1}^{k(n)} b_{n,i} > 0$ with $b_{n,i} \geqslant 0$ for all $1 \leqslant i \leqslant k(n)$. Let

$$T_n := \sum_{i=1}^{k(n)} X_{n,i} \tag{5}$$

be our test statistic which is assumed to be tight under H_0 with non-constant weak accumulation points. Fix a level $\alpha \in (0,1)$. Consider for each $P \in \mathrm{H}_0$ tests ϕ_n with critical values $c_n = c_n(P)$

$$\mathbf{1}_{(c_n,\infty)}(T_n) \leqslant \phi_n \leqslant \mathbf{1}_{[c_n,\infty)}(T_n) \tag{6}$$