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Forward irrelevance☆

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ABSTRACT

We investigate how to combine marginal assessments about the values that random variables assume separately into a model for the values that they assume jointly, when (i) these marginal assessments are modelled by means of coherent lower previsions and (ii) we have the additional assumption that the random variables are forward epistemically irrelevant to each other. We consider and provide arguments for two possible combinations, namely the forward irrelevant natural extension and the forward irrelevant product, and we study the relationships between them. Our treatment also uncovers an interesting connection between the behavioural theory of coherent lower previsions, and Shafer and Vovk's game-theoretic approach to probability theory.

1. Introduction

In probability and statistics, assessments of independence are often useful as they allow us to reduce the complexity of inference problems. To give an example, and to set the stage for the developments in this paper, we consider two random variables X_1 and X_2, taking values in the respective *finite* sets $\mathscr{X}_1$ and $\mathscr{X}_2$. Suppose that a subject is uncertain about the values of these variables, but that he has some model expressing his beliefs about them. Then we say that X_1 is *epistemically irrelevant* to X_2 for the subject when he assesses that learning the actual value of X_1 would not change his beliefs (or belief model) about the value of X_2. We say that X_1 and X_2 are *epistemically independent* when X_1 and X_2 are epistemically irrelevant to one another; the terminology is borrowed from Walley (1991, Chapter 9).

Let us first look at what these general definitions yield when the belief models our subject uses are precise probabilities. If the subject has a marginal probability mass function $p_1(x_1)$ for the first variable X_1, and a conditional mass function $q_2(x_2|x_1)$ for the second variable X_2 conditional on the first, then we can calculate his joint mass function $p(x_1,x_2)$ using Bayes's rule: $p(x_1,x_2)=p_1(x_1)q_2(x_2|x_1)$. Now consider any real-valued function f on $\mathscr{X}_1\times\mathscr{X}_2$. We shall call such functions *gambles*, because they can be interpreted as uncertain rewards. We find for the prevision (or expectation, or fair price, we use de Finetti's (1974–1975)

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terminology and notation throughout this paper) of such a gamble f that

$$\begin{aligned} P(f) &= \sum_{(x_1,x_2)\in\mathscr{X}_1\times\mathscr{X}_2} f(x_1,x_2)p_1(x_1)q_2(x_2|x_1) \\ &= \sum_{x_1\in\mathscr{X}_1} p_1(x_1)\sum_{x_2\in\mathscr{X}_2} f(x_1,x_2)q_2(x_2|x_1) = \sum_{x_1\in\mathscr{X}_1} p_1(x_1)Q_2(f(x_1,\cdot)|x_1) \\ &= P_1(Q_2(f|X_1)), \end{aligned} \tag{1}$$

where we let $Q_2(f|X_1)$ be the subject's conditional prevision of f given X_1, which is a gamble on $\mathscr{X}_1$ whose value in x_1,

$$Q_2(f|x_1) := Q_2(f(x_1,\cdot)|x_1) = \sum_{x_2\in\mathscr{X}_2} f(x_1,x_2)q_2(x_2|x_1)$$

is the subject's conditional prevision of f given that $X_1 = x_1$. We also let P_1 be the subject's marginal prevision (operator) for the first random variable, associated with the marginal mass function p_1: $P_1(g) := \sum_{x_1\in\mathscr{X}_1} g(x_1)p_1(x_1)$ for all gambles g on $\mathscr{X}_1$.

When the subject judges X_1 to be (epistemically) irrelevant to X_2, then we get for all $x_1 \in \mathscr{X}_1$ and $x_2 \in \mathscr{X}_2$ that

$$q_2(x_2|x_1) = p_2(x_2), \tag{2}$$

where p_2 is the subject's marginal mass function for the second variable X_2 that we can derive from the joint p using $p_2(x_2) := \sum_{x_1\in\mathscr{X}_1} p(x_1,x_2)$. The equality (2) expresses that learning that $X_1 = x_1$ does not change the subject's probability model for the value of the second variable. Condition (2) is equivalent to requiring that for all $x_1 \in \mathscr{X}$ and all gambles f on $\mathscr{X}_1 \times \mathscr{X}_2$,

$$Q_2(f(x_1,\cdot)|x_1) = P_2(f(x_1,\cdot)), \tag{3}$$

where now P_2 is the subject's marginal prevision (operator) for the second variable, associated with the marginal mass function p_2. We can then write for the joint prevision:

$$P(f) = P_1(P_2(f)), \tag{4}$$

where f is any gamble on $\mathscr{X}_1 \times \mathscr{X}_2$, and where we let $P_2(f)$ be the gamble on $\mathscr{X}_1$ that assumes the value $P_2(f(x_1,\cdot))$ in $x_1 \in \mathscr{X}_1$.

Similarly, when X_2 is epistemically irrelevant to X_1 for our subject, then

$$q_1(x_1|x_2) = p_1(x_1) \tag{5}$$

for all $x_1 \in \mathscr{X}_1$ and $x_2 \in \mathscr{X}_2$. Here $q_1(x_1|x_2)$ is the subject's mass function for the first variable X_1 conditional on the second. This leads to another expression for the joint prevision:

$$P(f) = P_2(P_1(f)). \tag{6}$$

Expressions (4) and (6) for the joint are equivalent, as generally $P_1(P_2(f)) = P_2(P_1(f))$. This is related to the fact that conditions (2) and (5) are equivalent: if X_1 is epistemically irrelevant to X_2 then X_2 is epistemically irrelevant to X_1, and *vice versa*. In other words, *for precise probability models, epistemic irrelevance is equivalent to epistemic independence.*

Some caution is needed here: this equivalence is only guaranteed if the marginal mass functions are everywhere non-zero. If some events have zero probability, then it can still be guaranteed provided we slightly change the definition of epistemic irrelevance, and, for instance, impose $q_2(x_2|x_1) = p_2(x_2)$ only when $p_1(x_1) > 0$.

All of this will seem tritely obvious to anyone with a basic knowledge of probability theory, but the point we want to make, is that the situation changes dramatically when we use belief models that are more general (and arguably more realistic) than the precise (Bayesian) ones, such as Walley's (1991) imprecise probability models.

On Walley's view, a subject may not generally be disposed to specify a fair price $P(f)$ for any gamble f, but we can always ask for his *lower prevision* $\underline{P}(f)$, which is his supremum acceptable price for buying the uncertain reward f, and his *upper prevision* $\overline{P}(f)$, which is his infimum acceptable price for selling f. We give a fairly detailed introduction to Walley's theory in Section 2.

On this new approach, if X_1 is epistemically irrelevant to X_2 for our subject, then [compare with Condition (3)]

$$\underline{Q}_2(f(x_1,\cdot)|x_1) = \underline{P}_2(f(x_1,\cdot))$$

for all gambles f on $\mathscr{X}_1 \times \mathscr{X}_2$ and all $x_1 \in \mathscr{X}_1$. Here, similar to what we did before, $\underline{P}_2$ is the subject's marginal lower prevision (operator) for X_2, and $\underline{Q}_2(\cdot|X_1)$ is his lower prevision (operator) for X_2 *conditional* on X_1. We shall see in Section 3 that a reasonable joint model[1] for the value that (X_1, X_2) assumes in $\mathscr{X}_1 \times \mathscr{X}_2$ is then given by [compare with Eqs. (1) and (4)]

$$\underline{P}(f) = \underline{P}_1(\underline{Q}_2(f|X_1)) = \underline{P}_1(\underline{P}_2(f)) \tag{7}$$

for all gambles f on $\mathscr{X}_1 \times \mathscr{X}_2$, where $\underline{P}_1$ is the subject's marginal lower prevision (operator) for X_1, and where we also let $\underline{P}_2(f)$ be the gamble on $\mathscr{X}_1$ that assumes the value $\underline{P}_2(f(x_1,\cdot))$ in any $x_1 \in \mathscr{X}_1$.

[1] This is the most conservative joint lower prevision that is coherent with $\underline{P}_1$ and $\underline{Q}_2(\cdot|X_1)$, see also Walley (1991, Section 6.7).