Journal of Statistical Planning and Inference 139 (2009) 593 – 616

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Score test of fit for composite hypothesis in the GARCH(1, 1) model

Bartosz Stawiarski

Institute of Mathematics and Informatics, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

ARTICLE INFO

Article history:
Received 26 September 2006
Received in revised form
19 May 2007
Accepted 8 April 2008
Available online 22 May 2008

MSC:
62G10
62E20
65C05
37M10

Keywords:
GARCH(1, 1) model
Conditional distribution
Efficient score vector
Data-driven test of fit
Square-root consistent estimator
Martingale difference array
Central limit theorem
Monte Carlo simulations
GED family

ABSTRACT

A score test of fit for testing the conditional distribution of the stationary GARCH(1, 1) model conceived by Bollerslev [1986. Generalized autoregressive conditional heteroskedasticity. J. Econometrics 31, 307–327] is proposed. The null hypothesis asserting that the noise distribution belongs to the specified parametric class of distributions is considered.
Exploiting the pioneer idea of Neyman [1937. Smooth test for goodness of fit. Skand. Aktuarietidskr. 20, 149–199] and the device proposed by Ledwina [1994. Data driven version of Neyman's smooth test of fit. J. Amer. Stat. Assoc. 89, 1000–1005], the efficient score statistic and its data-driven version are derived for this testing problem. The asymptotic null distribution of the score statistic is established. Replacing the nuisance parameters with their square-root consistent estimators results in the data-driven test statistic. It is proved that in that case the asymptotic behaviour of the test statistic remains unchanged under appropriate regularity conditions and under discretization of the estimators. Computer simulations of the critical value and the power performance of the test for several alternatives in the case of generalized error distribution family serving as a null distribution are also presented.

1. Introduction

Conditionally normal GARCH models, introduced by Bollerslev (1986), were soon proved on an empirical basis to fit financial time series poorly, leading to considering non-Gaussian distributions, see e.g. Bollerslev (1987), Teräsvirta (1996), Mittnik et al. (1998). However, the problem of testing conditional distribution assumptions in dynamic models (including GARCH-type ones) has become a subject of more intensive research only in recent years. One motivation behind developing new testing tools is that the applicability of Kolmogorov–Smirnov or Cramer–von-Mises type tests in the dependent data framework accompanied by model parameters is seriously limited, as pointed out in Chen (2002).

Up to now, few tests designed for verification of the composite hypothesis concerning the conditional distribution in GARCH models have appeared. Chen (2002) considered this problem in the more general context of time series models with specified conditional mean and variance, $X_t = m_t(Z_t, \eta_1) + \sqrt{h_t(Z_t, \eta_2)}\varepsilon_t$, where an explanatory variable Z_t is allowed to be a lagged dependent variable and η_i stand for nuisance parameters. The Chen's characteristic-function-based test is also designed for the composite hypothesis context (to be clarified below) and is asymptotically distribution-free. However, its construction requires choosing suitable weight functions and applying a "static transformation" to improve its power. In Chen's research, conditionally t- and generalised error distribution (GED)-distributed GARCH models were often accepted as reliable in fitting some stock indices returns.

E-mail addresses: Bartosz.Stawiarski@pwr.wroc.pl, bjs13@wp.pl.

0378-3758/$ - see front matter
doi:10.1016/j.jspi.2008.04.033

The test proposed in Bai (2003) is based on a martingale Khmaladze transform and can be applied for testing conditional distribution of some dynamic models. Specifically, for instance, testing the noise distribution represented by a parametric scale family in the ARMA models can be performed. The testing solution in Bai (2003) is also devised for testing conditional distribution of GARCH models in the composite hypothesis framework. Still, as shown by simulations in Inglot and Stawiarski (2005), Bai's test works well enough whenever testing conditional GARCH normality is addressed but is nearly useless for the Laplace case.

Koul and Ling (2006) treat the problem of fitting the innovations distribution in the strictly stationary and ergodic time series as equivalent to testing a goodness-of-fit hypothesis in the conditional mean and variance formulation similar to that of Chen (2002). The test is applicable in the GARCH or ARMA–GARCH context, but the hypothesis is simple, i.e. the null density is fully specified. Construction of their test is based on the properly weighted empirical distribution function, so in its spirit it is the Kolmogorov-type tool. Besides, critical values depend on the hypothetical density and have to be simulated. The power study focuses on testing conditional normality of the AR(1)–GARCH(1,1) series, but the reported powers are lower than those obtained in Inglot and Stawiarski (2005).

Recently, the idea of smooth tests conceived by Neyman (1937) and their data-driven versions devised by Ledwina (1994) was reconsidered in the time series context. Ducharme and Lafaye de Micheaux (2004) derived the data-driven goodness-of-fit test of normality in causal and invertible ARMA models under known mean or trend. The hypothesis verified there is composite with the variance being an unknown nuisance parameter. The authors also suggested that the test can be employed for non-Gaussian hypotheses in the ARMA context with unknown location, scale and shape parameters. Accordingly, keeping in mind both very good performance of the data-driven tests of fit in *i.i.d.* cases, and the prominent role of GARCH models in fitting financial time series, it seems very promising and challenging to develop this tool in the conditionally heteroscedastic setting.

The score test of fit for verifying the conditional distribution of the GARCH(1, 1) model was addressed in Inglot and Stawiarski (2005). The data-driven score statistic and its asymptotic behaviour were derived for the case of simple hypothesis, i.e. when the conditional GARCH zero-mean and unit-variance distribution did not depend on additional nuisance parameters. To be more precise, the hypothesis considered in Inglot and Stawiarski (2005) was composite, too, due to the presence of the GARCH(1, 1) model (nuisance) parameter. By the "simple" hypothesis we mean the one concerning the fully specified noise distribution, as opposed to the "composite" case of a null parametric family of distributions considered in the present paper. In the previous paper, however, the asymptotic distribution of the proposed test statistic $\hat{W}_{\hat{S}}(\hat{\vartheta})$ (estimated by plugging-in the GARCH(1, 1) parameter estimator) was not established. Anyway, the simulation study enclosed in Section 4 of Inglot and Stawiarski (2005) quite convincingly confirmed stabilization of the estimated critical values with growing sample size n. Very good sensitivity of the test was also shown via empirical power simulations both for standard normal and Laplace null distributions.

In this paper we propose a score test of fit for composite hypothesis, treating jointly the GARCH(1, 1) model parameter and the parameter of hypothetical family of distributions as the nuisance parameter in our problem. The asymptotic distribution of the data-driven test statistic is established, encompassing the case of simple hypothesis from Inglot and Stawiarski (2005), too. Note that the problem of composite testing for the *i.i.d.* case with application of data-driven score test was considered in Inglot et al. (1997).

In Section 2 the problem of testing composite hypothesis is drawn up, together with basic model assumptions. In Section 3 the efficient score statistic together with its data-driven version based on a penalized dimension selection rule are derived. The asymptotic behaviour of the data-driven score statistic is also established under rather mild assumptions. Asymptotic distribution of the data-driven test statistic with estimated nuisance parameters is the subject of Section 4, where the main limit result is stated. Section 5 presents simulations of the test performance for the specific case of GED family. Section 6 contains proofs of theorems and propositions stated in Section 3. In Section 7 proofs of auxiliary lemmas are enclosed, paving the way for the proof of our main Theorem 4.1. This proof is an exclusive subject of Section 8. Finally, Section 9 deals with checking the assumptions for GED family of distributions considered in Section 5.

2. The testing problem

A GARCH(1, 1) time series $\{X_t\}_{t \in Z}$, introduced by Bollerslev (1986), is defined by the following relations:

$$\begin{cases} X_t = \sqrt{h_t}\varepsilon_t \\ h_t = \omega + \alpha X_{t-1}^2 + \beta h_{t-1} \end{cases}, \quad t \in Z = \{\dots, -2, -1, 0, 1, 2, \dots\}, \tag{2.1}$$

where $\vartheta = [\omega, \alpha, \beta]^{\mathrm{T}}$ is a column vector of model parameters and $\{\varepsilon_t\}_{t \in Z}$ is a sequence of *i.i.d.* random variables, satisfying $E\varepsilon_t = 0$, $Var\,\varepsilon_t = 1$. From now on we will assume that $\vartheta \in \Theta = \{[\omega, \alpha, \beta]^{\mathrm{T}} : \omega, \alpha, \beta > 0; \alpha + \beta < 1\}$, which ensures weak and strict stationarity as well as ergodicity of X_t (see e.g. Bollerslev, 1986; Nelson, 1990). A strictly stationary and ergodic solution for conditional variance h_t in general GARCH(p, q) case can be found e.g. in Li and Ling (1997). Denoting by $\mathscr{F}_t = \sigma\{\dots, \varepsilon_{t-1}, \varepsilon_t\}$ the σ-field of the process history up to time t it is clear that h_t is $\mathscr{F}_{t-1}$-measurable.

The null hypothesis in Inglot and Stawiarski (2005) asserted that the innovations ε_t have a fully specified unknown density $f(x)$ on the real line, with zero mean and unit variance. It is more natural and realistic, both for theoretical and practical reasons, to allow for some nuisance parameters in the density.