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An L-statistic approach to a test of exponentiality against IFR alternatives

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Abstract

In this note we consider the problem of testing exponentiality against IFR alternatives. A measure of deviation from exponentiality is developed and a test statistic constructed on the basis of this measure. It is shown that the test statistic is an L-statistic. The asymptotic as well as the exact distribution of the test statistic is obtained and the test is shown to be consistent.

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1. Introduction

The exponential distribution has been widely used in the theory of reliability and life testing. Typically, when one is dealing with failure times of items such as fuses, transistors, bulbs, etc., where failure is caused due to sudden shocks rather than wear and tear, the assumption of exponentiality is particularly justified. Testing for exponentiality of the failure time is, in effect, the same as testing whether the shocks responsible for the failure arrive according to a Poisson process.

Equally important in reliability theory is the concept of ageing. '*No ageing*' means the age of the component has no effect on the distribution of its residual lifetime. '*Positive ageing*' means that age has an adverse effect, in some probabilistic sense, on the residual lifetime. '*Negative ageing*' describes the opposite beneficial effect of ageing. If the same type of ageing persists throughout the entire lifetime of a unit, it is referred to as '*monotonic ageing*'.

In many physical situations the subject does become more vulnerable to failure with increasing age. This is characteristic of objects subject to wear out, e.g., moving parts, human beings past youth, etc. In such situations one would expect the failure distribution to be characterized by an increasing failure rate. Examples of such distributions are the Weibull distribution with density $f(x)=\lambda^{\alpha}\alpha t^{\alpha-1}e^{-(\lambda t)^{\alpha}}$, $t\geqslant 0$, $\alpha\geqslant 1$, $\lambda>0$, the gamma with density $f(t)=(e^{-\lambda t}\lambda^{\alpha}t^{\alpha-1})/\Gamma(\alpha)$, $t\geqslant 0$, $\alpha\geqslant 1$, $\lambda>0$ and the linear failure rate distribution with density $f(x)=e^{-(x+\theta x^2/2)}\{1+\theta x\}$.

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A life distribution can be classified according to the monotonicity properties of the failure rate. If the failure rate of a distribution increases monotonically, the distribution has increasing failure rate (IFR). The units with IFR deteriorates with age. Proschan and Pyke (1967) list several advantages if the failure rate is known to be increasing.

The exponential distribution is often taken as a benchmark in reliability theory. An assumption of exponentially distributed lifetimes implies that a used item is stochastically as good as new, so there is no reason to replace a functioning item. The exponential distribution is the most commonly used life distribution in applied probability analysis (mainly because of its mathematical simplicity). It is, therefore, of interest to detect possible departures from exponentiality in the data, such as IFR, for instance.

The problem of testing exponentiality against IFR alternatives is, therefore, an important problem in reliability literature and has been widely discussed. In fact, different approaches have been used to solve this testing problem. Proschan and Pyke (1967) proposed a test based on the ranks of the normalized spacings between the ordered observations. Bickel and Doksum (1969) investigated the asymptotic power behavior of various tests for this testing problem and showed the test proposed by Proschan and Pyke (1967) is asymptotically inadmissible. Other early papers include those of Bickel (1969), Bickel and Doksum (1969) and Barlow and Proschan (1969). Later, Barlow and Doksum (1972) proved that a test which rejects exponentiality in favor of IFR alternatives when the signed area between the TTT-plot and the diagonal is large is asymptotically minimax. Ahmad (1975, 1976, 2004) proposed non-parametric tests based on Hoeffding's (1948) U-statistic for the same problem. Deshpande and Kochar (1983) also proposed a test based on a U-statistic while Klefsjö (1983) based his test on the scaled total time on test transform. Later Aly (1990) derived a family of tests for the same problem by defining the IFR plot function and extended it to accommodate randomly censored data. Belzunce et al. (1998) and Muralidharan and Shanubhogue (2003) proposed tests based on stochastic ordering of random variables. Ahmad (2001) suggested a test based on moment inequality. Earlier Lai (1994) had made a survey of tests of univariate and bivariate stochastic ageing. The recent book by Lai and Xie (2006) provides a very useful panoramic view of theory and application of ageing and dependence in the use of mathematical methods in reliability and survival analysis.

We contribute to the literature by considering the same problem and proposing a new test procedure. Our test statistic is an L-statistic. We derive its exact distribution and show that the test is asymptotically normal. We also prove that it is consistent.

2. A measure of deviation

Let $\mathscr{E}$ be the class of exponential distributions, with distribution function $F(x) = 1 - e^{-\lambda x}, x \geqslant 0$ where λ is any positive number, typically unknown. Formally our problem is to test

$$H_0 : F \in \mathscr{E} \quad \text{vs} \quad H_1 : F \in \text{IFR} - \mathscr{E}$$

based on a random sample $X_1, X_2, \ldots, X_n$ of size n from an absolutely continuous distribution F with density f and survival function $\overline{F}$.

Define the hazard rate $r(t) := f(t)/\overline{F}(t)$. Consider the functional of F given by

$$T(F) = \iint [r(t) - r(s)]\overline{F}(s)\overline{F}(t)\,\mathrm{d}s\,\mathrm{d}t,$$

where the integration is defined on the set $\{(s,t) : 0 \leqslant s \leqslant t < \infty\}$.

On simplifying $T(F)$ reduces to $2\int_0^\infty \{\overline{F}(t)\}^2\,\mathrm{d}t - \mu$, where μ is the mean of F.

Observe that under H_0, $T(F) = 0$; and under H_1, $T(F) > 0$, F being assumed continuous. Thus, $T(F)$ provides a measure of deviation from exponentiality towards IFR alternatives.

The sample analog of $T(F)$ forms our test statistic. Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denote the order statistics based on a random sample $X_1, X_2, \ldots, X_n$ and let F_n be the empirical cdf. The test statistic is

$$T_n \equiv T(F_n) = 2\int_0^\infty \{\overline{F}_n(t)\}^2\,\mathrm{d}t - \overline{X}_n.$$