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ScienceDirect
Journal of Statistical Planning and Inference 138 (2008) 3157–3163
journal of statistical planning and inference
www.elsevier.com/locate/jspi

Defining equations for two-level factorial designs☆

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Received 30 November 2007; received in revised form 4 January 2008; accepted 8 January 2008
Available online 17 January 2008

Abstract

Defining equations are introduced in the context of two-level factorial designs and they are shown to provide a concise specification of both regular and nonregular designs. The equations are used to find orthogonal arrays of high strength and some optimal designs. The latter optimal designs are formed in a new way by augmenting notional orthogonal arrays which are allowed to have some runs with a negative number of replicates before augmentation. Defining equations are also shown to be useful when the factorial design is blocked.

Keywords: Aliasing; Augmented design; Defining contrasts; Fractional factorial; J-characteristics; Orthogonal arrays

1. Introduction

Two-level factorial designs have been widely studied and some of the most notable papers in the literature over the last 30 years contain work on optimal designs (Cheng, 1980; Jacroux et al., 1983; Chadjiconstantinidis et al., 1989; Mukerjee, 1995; Butler, 2006, 2008), minimum aberration of regular designs (Fries and Hunter, 1980; Chen, 1992; Chen et al., 1993; Chen and Hedayat, 1996; Tang and Wu, 1996; Butler, 2003a) and minimum aberration of nonregular designs (Tang and Deng, 1999; Tang, 2001; Xu and Wu, 2001; Butler, 2003b).

Two-level factorial designs are often specified in terms of defining contrasts. These provide a concise specification of the experimental runs in the design and give a complete understanding of the aliasing structure of the design. Designs that can be specified with defining contrasts are said to be regular. Tang (2001) generalized the idea of defining contrasts to nonregular designs by showing that a design is uniquely specified by its so-called J-characteristics. A J-characteristic for a main effect or interaction is simply the degree of aliasing or non-orthogonality of the effect in the design.

In this paper, these ideas are extended and made more explicit by introducing the defining equation of two-level factorial designs. This is first described in Section 2 by use of examples and then expressed in matrix notation in Section 3. The defining equation is used to construct orthogonal arrays of high strength in Section 4 and to construct some optimal designs in Section 5. The actual optimal designs in Section 5 are formed by augmenting notional orthogonal arrays in which some factorial combinations are allowed to have a negative number of replicates before augmentation. This is a new idea for constructing optimal designs and follows previous work on augmenting orthogonal arrays in the aforementioned papers (Cheng, 1980; Jacroux et al., 1983; Chadjiconstantinidis et al., 1989; Mukerjee,

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0378-3758/$ - see front matter
doi:10.1016/j.jspi.2008.01.002

1995; Butler, 2008). Defining equations for blocked designs are considered in Section 6 and some concluding remarks are made in Section 7.

2. Defining equations

Consider a two-level factorial design in m factors with levels 1 and -1 in each factor. A design is completely specified by the number of replicates of each factorial combination. The defining equation takes the form

$$R = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_m=0}^{1} c(i_1, i_2, \ldots, i_m) F_1^{i_1} F_2^{i_2} \cdots F_m^{i_m}, \tag{1}$$

where R is the number of replicates of each of the 2^m possible factorial combinations and is required to be a non-negative integer. Also, F_i is the ith factor and $F_i^0 = I$, where I is the identity, so for example $F_1^1 F_2^0 F_3^1$ is the two-factor interaction $F_1 F_3$. The coefficients $c(i_1, i_2, \ldots, i_m)$ reveal the degree of aliasing of the effects $F_1^{i_1} F_2^{i_2} \ldots F_m^{i_m}$ for $(i_1, i_2, \ldots, i_m)$ non-zero, and $c(0, 0, \ldots, 0)$ is the fraction of the 2^m design. If $c(i_1, i_2, \ldots, i_m) = c(0, 0, \ldots, 0)$ the effect is fully aliased in the design, if $c(i_1, i_2, \ldots, i_m) = 0$ the effect is not aliased, and if $0 < c(i_1, i_2, \ldots, i_m) < c(0, 0, \ldots, 0)$ the effect is partially aliased. The link with Tang (2001) is that the J-characteristics $J(i_1, i_2, \ldots, i_m)$ satisfy $J(i_1, i_2, \ldots, i_m) = 2^m c(i_1, i_2, \ldots, i_m)$.

The defining Eq. (1) is best illustrated by way of examples. Consider the defining equation for a nonregular design in $m = 4$ factors

$$R = \tfrac{3}{4} I + \tfrac{1}{4} ABC + \tfrac{1}{4} ABD + \tfrac{1}{4} ACD + \tfrac{1}{4} BCD + \tfrac{1}{4} ABCD \tag{2}$$

for factors F_1, F_2, F_3, F_4 equal to A, B, C, D, respectively. Observe that the coefficient $c(0, 0, 0, 0)$ in front of I equals $3/4$, and hence the design is a three-quarter fraction design with $2^4 \times 3/4 = 12$ runs. The three-factor and four-factor interactions are one-third partially aliased in the design as their coefficients $c(i_1, i_2, i_3, i_4)$ in the equation are $1/3$ the coefficient of I. There is no aliasing of main effects and two-factor interactions. For the run $(+, +, +, +)$, ABC, ABD, ACD, BCD and $ABCD$ all equal 1. Therefore, from the defining Eq. (2), there are $R = 2$ replicates of this run in the design. Similarly, for the run $(-, -, -, -)$, ABC, ABD, ACD and BCD equal -1, whilst $ABCD$ equals 1, and so $R = 0$. It can be shown using methods described in Section 4 that $ABC + ABD + ACD + BCD + ABCD$ equals 5, 1 or -3 for every run. Consequently, from (2), the number of replicates R is a non-negative integer for every run, as required.

The defining equation for a regular 2^{-p} fraction of a factorial design with defining contrasts $W_1 = +$, $W_2 = +, \ldots, W_p = +$ is

$$\begin{aligned} R &= \frac{1}{2^p} \prod_{i=1}^{p} (I + W_i) \\ &= \frac{1}{2^p} \left(I + W_1 + W_2 + \cdots + \prod_{i=1}^{p} W_i \right). \end{aligned}$$

Observe that $R = 1$ replicate for runs in the 2^{-p} fraction and $R = 0$ otherwise. Note that the coefficient of I in the equation is 2^{-p} indicating a 2^{-p} fraction, and the coefficients of all defining contrasts and generalized interactions equal 2^{-p} also indicating that the effects are fully aliased. An example of a regular fractional factorial design is a 2^{8-2} design with defining contrasts $ABCDE = +$ and $DEFGH = +$. This has defining equation

$$\begin{aligned} R &= \tfrac{1}{4}(I + ABCDE)(I + DEFGH) \\ &= \tfrac{1}{4} I + \tfrac{1}{4} ABCDE + \tfrac{1}{4} DEFGH + \tfrac{1}{4} ABCFGH. \end{aligned}$$

Note that the generalized interaction appears in the equation as well as the defining contrasts by using standard multiplication methods for defining contrasts. If instead the 2^{8-2} design had defining contrasts $ABCDE = -$ and