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# Non-validity of affine α-resolvability in regular group divisible designs

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ABSTRACT

The concept of affine α-resolvability has been discussed for block designs in literature since 1942 for $\alpha=1$ and in particular since 1963 for $\alpha \geqslant 2$. Among group divisible (GD) designs, affine α-resolvable designs are known for both classes of singular GD and semi-regular GD designs. However, no example has been found for an affine α-resolvable regular GD design in literature. In this paper, the validity of such concept will be disproved for regular GD designs in general. Thus the regular GD design does not possess any property of the affine α-resolvability.
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## 1. Introduction

A block design with parameters $v = mn, b, r, k, \lambda_1, \lambda_2$ is called a group divisible (GD) design if the following conditions are satisfied:

(1) there are $v = mn$ treatments and the treatments can be divided into $m$ groups of $n$ treatments each, such that any two treatments in the same group are first associates and two treatments from different groups are second associates;
(2) there are $b$ blocks and each block contains exactly $k$ treatments;
(3) every treatment occurs at most once in a block;
(4) every treatment occurs in exactly $r$ blocks;
(5) any pair of treatments which are $i$th associates occurs in exactly $\lambda_i$ blocks, $i = 1, 2$.

The GD design is said to be singular when $r - \lambda_1 = 0$, and to be semi-regular when $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$. Furthermore, the GD design is said to be regular (R) when $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$ (see Raghavarao, 1988). When $\lambda_1 = \lambda_2$ in a GD design, the design is called a balanced incomplete block (BIB) design with parameters $v, b, r, k, \lambda\ (=\lambda_1 = \lambda_2)$.

A block design is said to be α-resolvable if its $b$ blocks can be grouped into $t$ resolution sets of $\beta$ blocks each such that every treatment appears in each resolution set precisely α times. So $b = \beta t$ and $r = \alpha t$. A 1-resolvable design is simply called a resolvable design and a relation $v = sk$ holds for some positive integer $s$. An α-resolvable block design is said to be affine α-resolvable if any pair of blocks belonging to the same resolution set contains $q_1$ treatments in common, whereas any pair of blocks belonging to different

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resolution sets contains $q_2$ treatments in common (Shrikhande and Raghavarao, 1963). It is clear that in an affine $\alpha$-resolvable block design $q_1 = k(\alpha-1)/(\beta-1)$ and $q_2 = \alpha k/\beta = k^2/v$. Note that when $\alpha = 1$, this definition of (affine) 1-resolvability coincides with that by Bose (1942).

Since the complement of an affine resolvable block design with $v = sk$ is an affine $(s-1)$-resolvable block design (cf. Kageyama, 1973), there are many affine $\alpha$-resolvable singular or semi-regular GD designs in literature, for example, see Clatworthy (1973). In this paper, it will be shown in Theorem 2.1 that there does not exist an affine $\alpha$-resolvable regular GD design for $\alpha \geqslant 1$. This means that a concept of the affine $\alpha$-resolvability is not valid for a regular GD design. This is the first paper to show the ineffectiveness of affine $\alpha$-resolvability in design theory, though some special non-existence conditions are known (cf. Raghavarao, 1988, Chapter 12). The author believes that the present observation is a very significant result in block designs.

## 2. Non-existence

Let $\boldsymbol{N}$ be the $v \times b$ incidence matrix of an affine $\alpha$-resolvable GD design with parameters $v = mn, b = \beta t, r = \alpha t, k, \lambda_1, \lambda_2$. Then the following can be shown.

**Lemma 2.1** (*cf. Raghavarao, 1988*). *The matrix $\boldsymbol{NN}'$ has eigenvalues $rk$, $r-\lambda_1$ and $rk - v\lambda_2$ with multiplicities 1, $m(n-1)$ and $m-1$, respectively.*

**Lemma 2.2.** *The matrix $\boldsymbol{N}'\boldsymbol{N}$ has eigenvalues $rk$, $k\{1-(\alpha-1)/(\beta-1)\}$ and 0, with multiplicities 1, $b-t$ and $t-1$, respectively.*

**Proof.** For the $v \times b$ incidence matrix $\boldsymbol{N}$ of an affine $\alpha$-resolvable GD design with parameters $v = mn, b = \beta t, r = \alpha t, k, \lambda_1, \lambda_2$, it follows that $\boldsymbol{N}'\boldsymbol{N} = \boldsymbol{I}_t \otimes \boldsymbol{A} + (\boldsymbol{J}_t - \boldsymbol{I}_t) \otimes \boldsymbol{B}$, where $\boldsymbol{A} = (k-q_1)\boldsymbol{I}_\beta + q_1\boldsymbol{J}_\beta$ and $\boldsymbol{B} = q_2\boldsymbol{J}_\beta$, where $\boldsymbol{I}_s$ is the identity matrix of order $s$, $\boldsymbol{J}_s$ is an $s \times s$ matrix whose all elements are one and $\otimes$ denotes the Kronecker product of two matrices. After some calculation, it holds that $\boldsymbol{N}'\boldsymbol{N}$ has eigenvalues $k + (\beta-1)q_1 + \beta(t-1)q_2$, $k - q_1$ and $k + (\beta-1)q_1 - \beta q_2$ with respective multiplicities 1, $(\beta-1)t = b-t$ and $t-1$. Now, since $q_1 = k(\alpha-1)/(\beta-1)$ and $q_2 = \alpha k/\beta = k^2/v$, it can be shown that $k + (\beta-1)q_1 + \beta(t-1)q_2 = rk$, $k - q_1 = k\{1-(\alpha-1)/(\beta-1)\}$ and $k + (\beta-1)q_1 - \beta q_2 = 0$. This completes the proof. □

**Lemma 2.3** (*cf. Lang, 1986*). *The matrices $\boldsymbol{XY}$ and $\boldsymbol{YX}$ have the same non-zero eigenvalues with the same multiplicities, where the matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ are of appropriate sizes.*

The main theorem is now stated below.

**Theorem 2.1.** *There does not exist an affine $\alpha$-resolvable RGD design for $\alpha \geqslant 1$.*

**Proof.** Since $r - \lambda_1 > 0$, $rk - v\lambda_2 > 0$ and $r - \lambda_1 \neq rk - v\lambda_2$ in a RGD design, if the design is affine $\alpha$-resolvable, then, by Lemmas 2.1 and 2.2, $\boldsymbol{NN}'$ has three distinct non-zero eigenvalues, whereas $\boldsymbol{N}'\boldsymbol{N}$ has two distinct non-zero eigenvalues. This contradicts Lemma 2.3. Hence it follows that there does not exist an affine $\alpha$-resolvable RGD design. This completes the proof. □

**Remark 2.1.** Non-existence of affine $\alpha$-resolvable block designs has been discussed in literature, for example, see Raghavarao (1988, Chapter 12). However, regarding affine $\alpha$-resolvable RGD designs, by use of the well-known Hasse–Minkowski invariant, any complete solution has not been obtained. Furthermore, a number-theoretic approach on integrality of design parameters by use of the relation $b = v + t - 1$ or $b = v + r - 1$ can give only partial answers, though an affine resolvable BIB design can be characterized by the same approach (see Raghavarao, 1988; pp. 71–72).

**Remark 2.2.** It is well known that the dual structure of an affine $\alpha$-resolvable design is equivalent to a semi-regular GD design. Thus, the claim in Section 2 may be also stated as follows: Assume that a design and its dual are both GD designs, then a design is regular if and only if its dual is regular.

## 3. Bounds

An improvement of bounds on the number of blocks will be made. At first, it is known (cf. Raghavarao, 1988, p. 61) that in an $\alpha$-resolvable RGD design with parameters $v, b, r = \alpha t, k, \lambda_1, \lambda_2$, an inequality $b \geqslant v + t - 1$ holds for $\alpha \geqslant 1$. As in the proof of Theorem 2.1, since non-zero eigenvalues have to get the same multiplicities by Lemma 2.3, it follows that in an affine $\alpha$-resolvable RGD design, $b = v + t - 1$ holds. However, as Theorem 2.1 shows, there does not exist an affine $\alpha$-resolvable RGD design. Hence it can be conjectured that in an $\alpha$-resolvable RGD design with $r = \alpha t$ and $\alpha \geqslant 1$, an inequality $b \geqslant v + t$ holds. This may show an improvement of the inequality $b \geqslant v + t - 1$ by one. All such designs listed in Clatworthy (1973) satisfy $b \geqslant v + t$. There are three resolvable RGD designs, R1, R44, R98, with $b = v + r$, in Clatworthy (1973). In particular, when $\alpha \geqslant 2$, $b \geqslant v + t + 1$ for existing $\alpha$-resolvable RGD designs.

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