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Practical small sample inference for single lag subset autoregressive models

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Abstract

We propose a method for saddlepoint approximating the distribution of estimators in single lag subset autoregressive models of order one. By viewing the estimator as the root of an appropriate estimating equation, the approach circumvents the difficulty inherent in more standard methods that require an explicit expression for the estimator to be available. Plots of the densities reveal that the distributions of the Burg and maximum likelihood estimators are nearly identical. We show that one possible reason for this is the fact that Burg enjoys the property of estimation equation optimality among a class of estimators expressible as a ratio of quadratic forms in normal random variables, which includes Yule–Walker and least squares. By inverting a two-sided hypothesis test, we show how small sample confidence intervals for the parameters can be constructed from the saddlepoint approximations. Simulation studies reveal that the resulting intervals generally outperform traditional ones based on asymptotics and have good robustness properties with respect to heavy-tailed and skewed innovations. The applicability of the models is illustrated by analyzing a longitudinal data set in a novel manner.

Keywords: Yule–Walker; Burg; Maximum likelihood; Estimating equation; Saddlepoint approximation; Confidence interval; Longitudinal data

1. Introduction

This paper considers approximations to distributions of various estimators of the parameters in a Gaussian autoregressive model of order p, AR(p), where the coefficients of the first $p-1$ lags are zero. We call these, single lag subset AR models of order p, henceforth abbreviated to SAR(p). Note that the usual AR(1) model is a special case, being a SAR(1). Primarily motivated by the desire to improve inference in small samples, and using a result of Daniels (1983), we show that saddlepoint approximations for the estimators can be easily constructed by viewing them as solutions of appropriate estimating equations. The main benefit of this approach lies in the fact that the estimating equation does not have to be solved for the estimator in question, an otherwise nontrivial task in the case of the maximum likelihood estimator (MLE). The approximations can be subsequently inverted to yield highly accurate confidence intervals for the parameters. For small samples, the resulting coverage probabilities and their lengths are generally superior to those obtained from first order asymptotics, the traditional way of constructing confidence intervals.

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A secondary motivation for our work is the observation that AR models fitted via Burg's method tend to exhibit consistently larger Gaussian likelihoods than those fitted via Yule–Walker's (YW) method (e.g. Brockwell and Davis, 2002, Section 5.1.2). Recent work by Brockwell et al. (2005) suggests this feature extends to multivariate subset AR models, and is accentuated by proximity of the roots of the AR polynomial to the unit circle. Comparing the distributions of the YW, Burg, and maximum likelihood (ML) estimators should provide further insight into their different finite-sample performances, and the question of whether or not the densities of the Burg and ML estimators are *closer* in some sense than those of YW and ML. When the data come from a Gaussian $\mathrm{SAR}(p)$ model, the YW and Burg estimators take the form of ratios of quadratic forms in normal random variables. Results concerning the closeness of the YW and Burg estimators to the MLEs are important, since the former are easily constructed and are frequently used as initial inputs in optimization algorithms that search for the latter. This is especially important in higher order AR models and multivariate settings.

There is a vast literature on approximating the distribution of estimators of the coefficient in an AR(1) model. Daniels (1956) derived saddlepoint approximations for the density of the Burg estimator. White (1961) computes asymptotic expansions for the mean and variance of the least squares and ML estimators. Phillips (1978) obtained the Edgeworth and saddlepoint approximations to the density of the least squares estimator (LSE). Durbin (1980) explored the approximate distribution of partial serial correlation coefficients, which included the YW estimator. Using Edgeworth approximations, Ochi (1983) obtained asymptotic expansions to terms of order n^{-1} for the distribution of the AR(1) version of the generalized coefficient estimator $\hat{\phi}(c_1, c_2)$ presented in the next section. Fujikoshi and Ochi (1984) used the same technique to approximate the distribution of the ML estimator. Liebermann (1994a, b) obtained saddlepoint approximations to the cumulative distribution and probability density functions of ratios of quadratic forms in normal random variables and the LSE, by direct contour integration of the inversion formula. Saddlepoint approximations to the cumulative distribution function of the LSE in an AR(1) are also derived by Wang (1992). More recently, and using the methods of Daniels (1954), Butler and Paolella (1998a, b) obtained saddlepoint approximations to general ratios of quadratic forms in normal random variables.

To the best of our knowledge, the general $\mathrm{SAR}(p)$ model has only been investigated by Chan (1989), who deals with asymptotic results in the nearly nonstationary case. In the particular (nonstationary) instance when the SAR coefficient is equal to unity, the resulting model is equivalent to seasonally differencing the series and has been called the *seasonal random walk* by Latour and Roy (1987). This branch of the literature dealing with inference for nonstationary AR(1)s and unit roots is particularly popular in econometrics, but is not our concern here; we are interested in the stationary case. Some of these results are obtained through asymptotic expansions, a related technique to saddlepoint approximations. We mention in passing the recent works of Perron (1996), Dufour and Kiviet (1998), and Kiviet and Phillips (2005). Noteworthy are also the papers of Andrews (1993) dealing with median-bias-corrected least squares estimation of a stationary and nonstationary (unit root) AR(1) in the presence of a polynomial time trend, and the very recent work of Broda et al. (2007) which extends this by considering other forms of bias adjustment to the LSE and incorporates the use of saddlepoint approximations for increased accuracy.

Our work differs and builds on the above cited references in several respects. First, we extend the saddlepoint approximations from the AR(1) to the more general $\mathrm{SAR}(p)$ case. Secondly, we approach the inference problem from the point of view estimating equations, allowing us to tackle more complex estimators like the MLE, which have not been dealt with before. The method we employ allows us to not only accurately approximate the distributions of the various parameter estimators, but to also invert them to produce confidence intervals with good properties. In the case of the LSE, the resulting confidence intervals are identical to those of Andrews (1993), but while he can only handle least-squares inference, our method requires only knowledge of the moment generating function of the estimating equation of which the desired estimator is the unique solution. Additionally we shed new light on the Burg estimator, showing it to be optimal in at least two respects, and find it to be robust to departures from normality with respect to heavy-tailed and skewed innovations. Lastly, we illustrate a novel application of the $\mathrm{SAR}(p)$ model, providing a simpler time series-based alternative to standard parametric longitudinal data modeling.

The rest of the paper is organized as follows. Section 2 introduces the YW, Burg, and ML estimators of the AR coefficient in a $\mathrm{SAR}(p)$ process. We show that all but the MLE are special cases of a generalized estimator, $\hat{\phi}(c_1, c_2)$, expressible as a ratio of quadratic forms in normal random variables. In Section 3, saddlepoint approximations to the probability distribution and density functions of the family of estimators $\{\hat{\phi}(c_1, c_2),\ c_1 \geqslant 0, c_2 \geqslant 0\}$ and the MLE are derived. Plots of the densities of the various estimators are given in Section 4, and an estimating equation optimality property is proved for Burg. Section 5 shows how small sample confidence intervals can be constructed from the