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Cumulants as iterated integrals

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ABSTRACT

A formula expressing cumulants in terms of iterated integrals of the distribution function is derived. It generalizes results of Jones and Balakrishnan who computed expressions for cumulants up to order 4.

1. Introduction

The expectation of a random variable can be computed in many ways. One method involving only the distribution function is obtained by careful partial integration and looks as follows:

$$\mathbf{E}X=\int_0^\infty (1-F(t))\,dt-\int_{-\infty}^0 F(t)\,dt$$

A similar formula holds for the variance, which can be written as the following double integral:

$$\mathrm{Var}(X)=2\iint_{-\infty<t_1<t_2<\infty} F(t_1)(1-F(t_2))\,dt_1\,dt_2 \tag{1.1}$$

Analogues of these formulas expressing the third and fourth cumulants (skewness and kurtosis) in terms of iterated integrals of the distribution function were computed some time ago by Jones and Balakrishnan (2002). The proof relied on *ad hoc* partial integration, see also Bassan et al. (1999), where similar formulas for mean differences are considered.

The aim of the present note is a generalization of these expressions to cumulants of arbitrary order, resulting in a formula resembling the well-known Möbius inversion formula, which expresses cumulants in terms of moments.

The paper is organized as follows. After a short introduction to cumulants in Section 2 and a review of partitions and shuffles in Section 3 we give two proofs of the main result. The first one using an elementary identity for the Volterra integral operator and Chen's shuffle formula for multiple integrals is contained in Section 4. In the concluding Section 5 we give another proof based on a formula for multivariate cumulants due to Block and Fang (1988).

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2. Moments and cumulants

Let X be a random variable with distribution function $F(x)=P(X\le x)=\int_{-\infty}^{x} dF(t)$. Its *moments* are the numbers

$$m_n=\mathbf{E}X^n=\int_{-\infty}^{\infty} t^n\, dF(t)$$

Under some assumptions the sequence of moments contains the complete information about the distribution of X. It can be collected in the exponential *moment generating function* (formal Fourier–Laplace transform)

$$\mathcal{F}_X(z)=\mathbf{E}e^{zX}=\sum_{n=0}^{\infty}\frac{m_n}{n!}z^n=1+\frac{m_1}{1!}z+\frac{m_2}{2!}z^2+\cdots$$

and the Taylor coefficients of the formal logarithm of the m.g.f.

$$\log\mathcal{F}_X(z)=\sum_{n=1}^{\infty}\frac{\kappa_n}{n!}z^n$$

are called the *cumulants*. The first two are the expectation $\kappa_1=m_1=\mathbf{E}X$ and the variance $\kappa_2=m_2-m_1^2=\operatorname{Var}X$. After rescaling the following two become the skewness $\kappa_3/\kappa_2^{3/2}$ and the kurtosis κ_4/κ_2^2.

The cumulants carry the same information as the moments but for many purposes in a better digestible form, e.g., after a translation the moments behave like

$$m_n(X+\tau)=\sum_{k=0}^{n}\binom{n}{k}\tau^{n-k}m_k(X)$$

while the cumulants are

$$\kappa_n(X+\tau)=\begin{cases}\tau+\kappa_1(X), & n=1\\ \kappa_n(X), & n\ge 2\end{cases} \tag{2.1}$$

For this reason the cumulants are sometimes called the *semi-invariants* of X. The most important property of the cumulants is the identity

$$\kappa_n(X+Y)=\kappa_n(X)+\kappa_n(Y)$$

if X and Y are independent.

3. Partitions and shuffles

There is also a combinatorial formula expressing the cumulants as polynomials in the moments. A *set partition* of order n is a set $\pi=\{B_1,B_2,\ldots,B_p\}$ of disjoint subsets $B_i\subseteq\{1,2,\ldots,n\}$, called *blocks*, whose union is $\{1,2,\ldots,n\}$. Denote by $|\pi|$ the number of blocks of a partition π and by Π_n the set of all n-set partitions. It is a lattice under the refinement order

$$\pi\le\sigma \iff \text{every block of } \pi \text{ is contained in a block of } \sigma$$

with minimal element $\hat{0}_n=\{\{1\},\{2\},\ldots,\{n\}\}$ and maximal element $\hat{1}_n=\{\{1,2,\ldots,n\}\}$. For the combinatorial identities below we will employ the following conventions. A *partition* of the number n is an unordered sequence $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ such that $\sum\lambda_i=n$. The fact that λ is a partition of n is denoted by the symbol $\lambda\vdash n$. Most of the time it is convenient to collect equal summands and write a partition in the form

$$\lambda=1^{k_1}2^{k_2}\cdots n^{k_n}\vdash n \tag{3.1}$$

With these conventions we abbreviate $\lambda!=1!^{k_1}2!^{k_2}\cdots n!^{k_n}$.

Every set partition $\pi\in\Pi_n$ determines a number partition $\lambda=\lambda(\pi)\vdash n$ of the form (3.1), called its *type*, where k_j is the number of blocks $B\in\pi$ of size $|B|=j$. Given a partition (3.1) and a sequence $(a_n)_{n\in\mathbf{N}}$ of numbers we denote

$$a_\lambda=\prod a_j^{k_j}$$

similarly for a set partition π we let

$$a_\pi=a_{\lambda(\pi)}=\prod_{B\in\pi}a_{|B|}$$

For a fixed partition $\lambda\vdash n$, the number of set partitions $\pi\in\Pi_n$ with $\lambda(\pi)=\lambda$ is equal to the *Faa di Bruno coefficient*

$$\left\{\begin{matrix} n\\ \lambda\end{matrix}\right\}=\{\pi:\pi\sim\lambda\}=\frac{n!}{1!^{k_1}2!^{k_2}\cdots n!^{k_n}k_1!k_2!\cdots k_n!} \tag{3.2}$$

The well known moment–cumulant formula says

$$\kappa_n=\sum_{\pi\in\Pi_n}m_\pi\mu(\pi,\hat{1}_n) \tag{3.3}$$