Journal of Statistical Planning and Inference 141 (2011) 418–428

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

The tail behavior of the convolutions of Gamma random variables

Subhash Kochar [a], Maochao Xu [b,*]

[a] Department of Mathematics and Statistics, Portland State University, Portland, OR, USA
[b] Department of Mathematics, Illinois State University, Normal, IL, USA

ARTICLE INFO

Article history:
Received 1 February 2010
Received in revised form
10 June 2010
Accepted 16 June 2010
Available online 20 June 2010

Keywords:
Convolution
Dispersive order
Majorization
Right spread order
Skewness
Star order

ABSTRACT

Two sufficient conditions for comparing convolutions of heterogeneous gamma random variables in terms of star order are established. It is further shown that if the scale parameters of heterogeneous gamma random variables are more dispersed in the sense of majorization, then the convolutions are more dispersed according to the right spread order, which generalizes and strengthens the results in Diaconis and Perlman (1987), Kochar and Xu (2010) and Zhao and Balakrishnan (2009).

1. Introduction

The convolution of independent random variables has attracted considerable attention in the literature due to its typical applications in many applied areas. For example, in reliability theory, it is used to study the lifetimes of redundant standby systems with independent components (cf. Bon and Păltănea, 1999); in actuarial science, it is used to model the total claims on a number of policies in the individual risk model (cf. Kaas et al., 2001); in nonparametric goodness-of-fit tests, the limiting distributions of U-statistics are the convolutions of independent random variables (cf. Serfling, 1980, Section 5.2).

The gamma distribution is one of the most popular distributions in statistics, engineering and reliability applications. In particular, gamma distribution plays a prominent role in actuarial science since most total insurance claim distributions have roughly the same shape as gamma distributions: skewed to the right, nonnegatively supported and unimodal (cf. Furman, 2008). As is well known, the gamma distribution includes exponential and chi-square, two important distributions, as special cases. Due to the complicated nature of the distribution function of gamma random variable, most of the work in the literature discusses only the convolutions of exponential random variables. Some relevant references are Boland et al. (1994), Kochar and Ma (1999), Bon and Păltănea (1999), Zhao and Balakrishnan (2009) and Kochar and Xu (2010). Bock et al. (1987) and Diaconis and Perlman (1987) studied convolutions of gamma random variables.

Let $X_1,\ldots,X_n$ be a random sample from a gamma distribution with shape parameter $a>0$, scale parameter $\lambda>0$ and with density function

$$f(x)=\frac{\lambda^a}{\Gamma(a)}x^{a-1}\exp\{-\lambda x\},\quad x\geq 0.$$

* Corresponding author.
E-mail address: mxu2@ilstu.edu (M. Xu).

0378-3758/$ - see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.jspi.2010.06.019

We are interested in studying the stochastic properties of statistics of the form

$$W=\theta_1 X_1+\theta_2 X_2+\cdots+\theta_n X_n,$$

where $\theta_1,\ldots,\theta_n$ are positive weights (constants). Bock et al. (1987) showed that for n=2, if

$$t\le\frac{a(\theta_1+\theta_2)}{\lambda},$$

then $P(W\le t)$ is Schur-convex in (θ_1,θ_2); and if

$$t\ge\frac{(a+1/2)(\theta_1+\theta_2)}{\lambda},$$

then $P(W\ge t)$ is Schur-convex in (θ_1,θ_2). For general $n>2$, $P(W\le t)$ is Schur-convex in the region

$$\left\{\boldsymbol{\theta}:\min_{1\le i\le n}\theta_i\ge\frac{t\lambda}{na+1}\right\},$$

where $\boldsymbol{\theta}$ is the vector of $(\theta_1,\ldots,\theta_n)$, and $P(W\ge t)$ is Schur-convex in $\boldsymbol{\theta}$ for

$$t\ge\frac{(na+1)(\theta_1+\theta_2+\cdots+\theta_n)}{\lambda}.$$

Diaconis and Perlman (1987) further studied the tail probabilities of convolution of gamma random variables. They pointed out that if

$$(\theta_1,\ldots,\theta_n)\overset{m}{\succeq}(\theta'_1,\ldots,\theta'_n) \tag{1.1}$$

then

$$\mathrm{Var}\left(\sum_{i=1}^n\theta_i X_i\right)\ge\mathrm{Var}\left(\sum_{i=1}^n\theta'_i X_i\right),$$

where $\overset{m}{\succeq}$ means the majorization order (see Definition 2.5).

This property states that if the weights are more dispersed in the sense of majorization, then the convolutions are more dispersed about their means as measured by their variances. Diaconis and Perlman (1987) also wondered whether $\sum_{i=1}^n\theta_i X_i$ is more dispersed than $\sum_{i=1}^n\theta'_i X_i$ as measured by the stronger criterion of their tail probabilities. They tried to answer this question by proving that under the condition (1.1), the distribution functions of $\sum_{i=1}^n\theta_i X_i$ and $\sum_{i=1}^n\theta'_i X_i$ have only one crossing. However, they only proved this result for n=2. For $n\ge3$, they required further restrictions. Hence, this problem has been open for a long time, which is also known as *Unique Crossing Conjecture*, or UCC.

It is possible to investigate this problem in other metrics for the θ_i's. For instance, if X_{λ_i} are independent gamma random variables with a common shape parameter a and scale parameters λ_i and $X_{\lambda'_i}$ are independent gamma random variables with a common shape parameter a and scale parameters λ'_i, for i=1,…,n, Korwar (2002) showed that under the condition of majorization order, for $a\ge1$,

$$(\lambda_1,\ldots,\lambda_n)\overset{m}{\succeq}(\lambda'_1,\ldots,\lambda'_n)\implies\sum_{i=1}^n X_{\lambda_i}\ge_{disp}\sum_{i=1}^n X_{\lambda'_i},$$

where $\ge_{disp}$ means the dispersive order (see Definition 2.3). Khaledi and Kochar (2004) relaxed the condition by proving that under the p-larger order, which is a weaker order than the majorization order, for $a\ge1$,

$$(\lambda_1,\ldots,\lambda_n)\overset{p}{\succeq}(\lambda'_1,\ldots,\lambda'_n)\implies\sum_{i=1}^n X_{\lambda_i}\ge_{disp}\sum_{i=1}^n X_{\lambda'_i},$$

where $\overset{p}{\succeq}$ means the p-larger order (see Definition 2.7).

In this paper, we further investigate the problem posed by Diaconis and Perlman (1987). For the case of n=2, we strengthen the single crossing property to the star order. We also give a different sufficient condition for the star order to hold between two convolutions of scaled gamma random variables. It will be shown that

$$(\lambda_1,\lambda_2)\overset{m}{\succeq}(\lambda'_1,\lambda'_2)\implies X_{\lambda_1}+X_{\lambda_2}\ge_\star X_{\lambda'_1}+X_{\lambda'_2},$$

and also

$$(\theta_1,\theta_2)\overset{m}{\succeq}(\theta'_1,\theta'_2)\implies\theta_1X_1+\theta_2X_2\ge_\star\theta'_1X_1+\theta'_2X_2, \tag{1.2}$$

where X_1 and X_2 are independent and identically distributed gamma random variables, and $\ge_\star$ means the star order (see Definition 2.1). Note that (1.2) could be equivalently expressed as

$$(1/\lambda_1,1/\lambda_2)\overset{m}{\succeq}(1/\lambda'_1,1/\lambda'_2)\implies X_{\lambda_1}+X_{\lambda_2}\ge_\star X_{\lambda'_1}+X_{\lambda'_2}.$$