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Three variable full orthogonal designs of order 56

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Abstract

Using the doubling lemma, amicable sets of eight circulant matrices and four complementary negacyclic matrices, we show that all full triples are types of orthogonal designs of order 56. This implies that all full orthogonal designs $\mathrm{OD}(2^t 7; x, y, 2^t 7 - x - y)$ exist for any $t \geqslant 3$.

MSC: 05B20

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1. Introduction

An *orthogonal design* A of order n and type $(s_1, s_2, \ldots, s_u)$, in the commuting variables $\pm x_1, \pm x_2, \ldots, \pm x_u$, denoted $\mathrm{OD}(n; s_1, s_2, \ldots, s_u)$, is a square matrix of order n with entries $\pm x_k$ or 0, where each x_k occurs s_k times in each row and column such that distinct rows are pairwise orthogonal. In other words,

$$AA^{\mathrm{T}} = (s_1 x_1^2 + \cdots + s_u x_u^2) I_n,$$

where I_n is the identity matrix of order n. It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, defined as $\rho(n) = 8c + 2^d$, where $n = 2^a b$, b odd, $a = 4c + d$ and $0 \leqslant d < 4$.

It is conjectured that all full 3-tuples are types of orthogonal designs of order $8n$, see Geramita and Seberry (1979). This conjecture has been verified for $n \leqslant 6$ (Geramita and Seberry, 1979; Eades, 1977; Eades and Seberry Wallis, 1976; Holzmann and Kharaghani, 2000a, 2001; Holzmann et al., 2005). In a recent paper (Georgiou et al., 2002), a number of full orthogonal designs of order 56 are found and 82 triples are labeled as missing. In this paper, we will show that the conjecture also holds for $n = 7$. Our approach is to first generate all possible full triples in order 56 using the doubling lemma. To do so, we apply the doubling lemma to all possible sets of four complementary circulant matrices of order

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seven in four variables. Then, we search to find as many as possible sets of amicable circulant matrices of order 7 in eight or less variables and identify all those which are special amicable. In doing so, we are able to resolve a total of 224 triples out of 261 possible full triples. While the use of circulant matrices in constructing orthogonal designs is standard, it was shown in Holzmann et al. (2005), Holzmann et al. (to appear) that negacyclic matrices may be used for those cases where the circulant matrices either do not exist or are hard to find. We will use this method to construct the 37 missing orthogonal designs in order 56. We have the following results.

Theorem 1. *All* 261 *possible full orthogonal designs of order* 56 *in three variables exist*.

Corollary 1. *All full orthogonal designs* $\mathrm{OD}(2^t 7; x, y, 2^t 7 - x - y)$ *exist for any* $t \geqslant 3$.

The complete set of 261 orthogonal designs of order 56 in three variables is available at the website http://www.cs.uleth.ca/OD56triples.

2. Amicable sets of circulant matrices

Two orthogonal designs X and Y are called *amicable* if $XY^t = YX^t$. A set $\{A_1, A_2, \ldots, A_{2n}\}$ of square real matrices is said to be *amicable* if

$$\sum_{i=1}^{n} (A_{\sigma(2i-1)} A^t_{\sigma(2i)} - A_{\sigma(2i)} A^t_{\sigma(2i-1)}) = 0,$$

for some permutation σ of the set $\{1, 2, \ldots, 2n\}$. In this case, we say that $\{A_{\sigma(2i-1)}\}_{i=1}^n$ *matches* with $\{A_{\sigma(2i)}\}_{i=1}^n$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. Clearly, a set of mutually amicable matrices is amicable, but the converse is not true in general. A set of matrices $\{A_1, A_2, \ldots, A_n\}$ is said to satisfy an additive property, if the matrix $\sum_{i=1}^n A_i A_i^t$ is a multiple of the identity matrix.

Let $A = \{A_1, A_3, A_5, A_7\}$ and $B = \{A_2, A_4, A_6, A_8\}$ be two sets of additive circulant matrices in variables $\pm x_1, \pm x_2, \ldots, \pm x_u$. We say that there is a special amicable matching of type $(r_1 + s_1, r_2 + s_2, \ldots, r_u + s_u)$, where the sums are a formal ones, if

$$\sum_{i=1}^{4} (A_{2i-1} A^t_{2i} - A_{2i} A^t_{2i-1}) = 0,$$

and each x_k appears occurs r_k times in A and s_k times in B (over a fixed row or column index). For convenience when listing types, we follow the convention that a subscript indicates the multiplicity of the subscripted component. For example, $((7+0)_4, (0+5)_3)$ indicates there are four variables each occurring 7 times in A, and three other variables each occurring 5 times in B. Following Holzmann and Kharaghani (2000b), we also called such a set of matrices a *special amicable set*.

An amicable set of eight circulant matrices $\{A_1, A_2, \ldots, A_8\}$ of order n satisfying an additive property can be used in the Kharaghani array (Kharaghani, 2000):

$$K = \begin{pmatrix} A_1 & A_2 & A_4R & A_3R & A_6R & A_5R & A_8R & A_7R \\ -A_2 & A_1 & A_3R & -A_4R & A_5R & -A_6R & A_7R & -A_8R \\ -A_4R & -A_3R & A_1 & A_2 & -A_8^tR & A_7^tR & A_6^tR & -A_5^tR \\ -A_3R & A_4R & -A_2 & A_1 & A_7^tR & A_8^tR & -A_5^tR & -A_6^tR \\ -A_6R & -A_5R & A_8^tR & -A_7^tR & A_1 & A_2 & -A_4^tR & A_3^tR \\ -A_5R & A_6R & -A_7^tR & -A_8^tR & -A_2 & A_1 & A_3^tR & A_4^tR \\ -A_8R & -A_7R & -A_6^tR & A_5^tR & A_4^tR & -A_3^tR & A_1 & A_2 \\ -A_7R & A_8R & A_5^tR & A_6^tR & -A_3^tR & -A_4^tR & -A_2 & A_1 \end{pmatrix},$$

where R is the back-identity matrix of order n, to obtain an orthogonal matrix of order $8n$.