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Journal of Statistical Planning and Inference 137 (2007) 1424 – 1432
journal of statistical planning and inference
www.elsevier.com/locate/jspi

Multivariate Matsumoto–Yor property is rather restrictive

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Received 31 October 2005; received in revised form 25 February 2006; accepted 26 February 2006
Available online 11 May 2006

Abstract

Matsumoto and Yor [2001. An analogue of Pitman's $2M - X$ theorem for exponential Wiener functionals. Part II: the role of the GIG laws. Nagoya Math. J. 162, 65–86] discovered an interesting invariance property of a product of the generalized inverse Gaussian (GIG) and the gamma distributions. For univariate random variables or symmetric positive definite random matrices it is a characteristic property for this pair of distributions. It appears that for random vectors the Matsumoto–Yor property characterizes only very special families of multivariate GIG and gamma distributions: components of the respective random vectors are grouped into independent subvectors, each subvector having linearly dependent components. This complements the version of the multivariate Matsumoto–Yor property on trees and related characterization obtained in Massam and Wesołowski [2004. The Matsumoto–Yor property on trees. Bernoulli 10, 685–700].

MSC: 62H05; 62E10; 60E05

Keywords: Gamma distribution; Generalized inverse Gaussian distribution; Matsumoto–Yor property; Independence

1. Introduction

Consider two random variables: X having the generalized inverse Gaussian (GIG) distribution $\mu_{-p,a,a}$ and Y having the gamma distribution $\gamma_{p,a}$. Recall, that the GIG distribution $\mu_{-p,a,b}$, where $p \in R,\ a, b > 0$ are the parameters, is defined by

$$\mu_{-p,a,b}(\mathrm{d}x) = K_1 x^{-p-1} \exp(-a^{-1}x - (bx)^{-1}) I_{(0,\infty)}(x)\,\mathrm{d}x$$

and the gamma distribution $\gamma_{q,c}$, where $q, c > 0$ are parameters, is defined by

$$\gamma_{q,c}(\mathrm{d}y) = K_2 y^{q-1} \exp(-c^{-1}y) I_{(0,\infty)}(y)\,\mathrm{d}y$$

(K_1, K_2 are normalizing constants). Matsumoto and Yor (2001) observed that if random variables X and Y are independent then $U = (X + Y)^{-1}$ and $V = X^{-1} - (X + Y)^{-1}$ are also independent and have the same distributions as X and Y,

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0378-3758/$ - see front matter
doi:10.1016/j.jspi.2006.02.012

respectively. The following extension of the Matsumoto–Yor (MY in the sequel) property is immediate: if (X, Y) has the distribution $\mu_{-p,a,b} \otimes \gamma_{p,a}$ ($\otimes$ denotes the product measure) then (U, V) is distributed according to $\mu_{-p,b,a} \otimes \gamma_{p,b}$. Its interpretation in terms of Brownian motion and related stochastic processes was given in Matsumoto and Yor (2003). Letac and Wesołowski (2000) obtained its converse with the proof based on an application of the Laplace transform technique (see Wesołowski, 2002b for an alternative approach based on densities). They dealt additionally with a matrix variate version of the MY property deriving a related characterization under the assumption that densities of random matrices X and Y exist and are strictly positive real functions of the class C_2. Wesołowski (2002a) extended this result by weakening the smoothness assumption, imposed on densities, to differentiability. Regression-type characterizations obtained in that paper have been recently refined in Chou and Huang (2004). The MY property for random matrices of different dimensions and related characterization has been studied in Massam and Wesołowski (2006), where a connection with the conditional structure of Wishart matrices was strongly emphasized.

The bivariate version of the MY property and a respective characterization has been considered recently in Bobecka and Wesołowski (2005). It has been proved there that in the bivariate setting, the property does not characterize general families of bivariate GIG and gamma distributions, but is more restrictive: it implies that respective random vectors have independent or linearly dependent components.

Here we are concerned with the MY property for random vectors of higher dimensions. The bivariate case is only a starting point of our induction argument. It appears that random vectors with the MY property have to have components grouped into independent subvectors, each vector having linearly dependent GIG or gamma components, respectively. So, similarly as in two dimensions only very special types of multivariate GIG and gamma distributions are involved. In the proof we borrow a lot from Bobecka and Wesołowski (2004), which will be referred to as BW in the sequel.

It has to be emphasized that another version of the multivariate MY property, defined in the language of directed trees, has been studied recently in Massam and Wesołowski (2004). These authors showed that the MY-like independence properties (defined through taking different roots in the given undirected tree) characterize multivariate distributions $W(q, K_G, a)$ with densities

$$f(k) \propto |\mathbf{k}|^{q-1} \mathrm{e}^{-(a,k)}, \quad k \in M(G, K_G),$$

where $G = (V, E)$ is an undirected tree with p vertices (V is the set of vertices and E the set of vertices),

$$K_G = \{k_{i,j} = k_{j,i} \neq 0, (i, j) \in E, k_{i,j} = k_{j,i} = 0, (i, j) \notin E\},$$

$$M(G, K_G) = \{k = (k_1, \ldots, k_p) \in \mathbf{R}^p : \mathbf{k} = [k_{i,j}] \in \Omega_p^+, k_{i,i} = k_i, k_{i,j} \in K_G, i \neq j\},$$

where Ω_p^+ is the cone of $p \times p$ positive definite symmetric matrices. Such distributions, due to the shape of their density function, can be regarded as versions of a multivariate gamma law. However, their univariate marginals are not of the gamma type, moreover those attached to the leaves of the tree are independent GIGs.

Let us note that the development of studies related to the MY property is somehow parallel to investigations concerning the Lukacs characterization of the gamma law. The Lukacs theorem (1955) for the univariate case was followed by its matrix variate analogue—see Olkin and Rubin (1962), Casalis and Letac (1996), Letac and Massam (1998), and Bobecka and Wesołowski (2002). The case of random vectors, first studied in the bivariate case of constant regressions by Wang (1981), only recently has been considerably expanded—see Bobecka (2002), Bobecka and Wesołowski (2003) and BW.

2. Characterization

In the sequel, we will use the following definitions:

We will say that a positive random vector $\bar{Y} = (Y_1, \ldots, Y_n)$ has a distribution $MG^*(\bar{A}, \bar{p}, \bar{\lambda})$, where $\bar{A} = (A_1, \ldots, A_r)$ is such that $\bigcup_{i=1}^r A_i = \{1, \ldots, n\}$, $A_i \cap A_j = \emptyset$, $i \neq j$ and $\bar{p} = (p_1, \ldots, p_r)$, $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$, where $p_i, \lambda_j > 0$, if a Laplace transform of $\bar{Y}$ is of the form

$$L_{\bar{Y}}(\sigma_1, \ldots, \sigma_n) = \prod_{i=1}^r \left(1 - \sum_{j \in A_i} \lambda_j \sigma_j\right)^{-p_i},$$