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Bayesian and likelihood-based inference for the bivariate normal correlation coefficient

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ABSTRACT

The paper develops some objective priors for correlation coefficient of the bivariate normal distribution. The criterion used is the asymptotic matching of coverage probabilities of Bayesian credible intervals with the corresponding frequentist coverage probabilities. The paper uses various matching criteria, namely, quantile matching, highest posterior density matching, and matching via inversion of test statistics. Each matching criterion leads to a different prior for the parameter of interest. We evaluate their performance by comparing credible intervals through simulation studies. In addition, inference through several likelihood-based methods have been discussed.

1. Introduction

Inference for the correlation coefficient of a bivariate normal distribution spans over several decades. Beginning with Fisher's derivation of the exact distribution of the sample correlation coefficient from a bivariate normal distribution as well as his celebrated hyperbolic tangent transformation, there have been multiple approaches towards this very important and practical problem of interest. More importantly, this is one situation where there is potential for a synthesis of the Bayesian, fiducial and likelihood-based inference.

'Objective' or 'default' priors are particularly suitable in this regard. However, not too surprisingly, over the years, the catalog of such priors has become prohibitively large, and it is important to specify some criterion for the selection of such priors.

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One such criterion which has found some appeal to both frequentists and Bayesians is the so-called 'probability matching' criterion. Simply put, this amounts to the requirement that the coverage probability of a Bayesian credible region is asymptotically equivalent to the coverage probability of the corresponding frequentist confidence region upto a certain order. An excellent monograph on this topic is due to Datta and Mukerjee (2004) which provides a thorough and comprehensive discussion of various probability matching criteria. Other review papers are due to Kass and Wasserman (1996), Ghosh and Mukerjee (1998) and Datta and Sweeting (2005).

The objective of this article is to find different priors according to several probability matching criteria, namely (a) posterior quantiles, (b) distribution functions, (c) highest posterior density (HPD) regions or (d) inversion of test statistics, when the bivariate normal correlation coefficient is the parameter of interest. We compare their performance for moderate sample sizes. In addition, we have considered several likelihood-based methods as well for similar inferential purposes.

The outline of the remaining sections is as follows. In Section 2 of this paper, we have introduced an orthogonal reparameterization of the bivariate normal parameters which greatly facilitates both Bayesian and likelihood-based inference. Section 3 develops a class of quantile matching priors, whereas Section 4 develops a general class of HPD Matching priors. Matching priors based on inversion of likelihood-ratio test statistics are developed in Section 5. There however, does not exist a prior that satisfies the matching via distribution functions criterion. Section 6 undertakes a simulation study and compares various credible intervals based on different priors. Section 7 develops inference based on conditional profile likelihood, adjusted profile likelihood and integrated likelihood. Some final remarks are made in Section 8.

In a recent interesting article, Berger and Sun (2008) have considered quantile matching priors for a variety of parameters evolving from the bivariate normal distribution. One important example of theirs is the bivariate normal correlation coefficient. However, these authors have not considered other matching criteria. While quantile matching property is quite desirable for one-sided credible intervals, the HPD matching or matching via inversion of test statistics seems more appropriate for two-sided credible intervals. All these priors also have intuitive appeal in the development of integrated likelihoods.

Matching priors for the ratio of variances in a bivariate normal distribution are given in Ghosh et al. (2009). Ghosh et al. (2008b) found priors satisfying all the four matching criteria (a)–(d) simultaneously for several parameters of interest, namely, the regression coefficients, generalized variances and the ratio of the conditional variance of one variable given the other to its marginal variance.

2. The orthogonal parameterization

Let $(X_{1i}, X_{2i}), (i=1,\ldots,n)$ be independent and identically distributed random variables having a bivariate normal distribution with means zero, variances $\sigma_1^2(>0)$ and $\sigma_2^2(>0)$, and correlation coefficient $\rho(|\rho|<1)$. The assumption of zero means is made to simplify notations and does not entail any lack of generalizability, since otherwise the posterior inference for ρ will only involve one less of degree of freedom, everything else remaining unchanged. Using the transformation

$$\theta_1=\sigma_1/\sigma_2, \theta_2=\sigma_1\sigma_2(1-\rho^2)^{1/2} \quad \text{and} \quad \theta_3=\rho, \tag{1}$$

the bivariate normal pdf can be rewritten as

$$f(X_1,X_2|\theta_1,\theta_2,\theta_3)\propto \frac{1}{\theta_2}\exp\left\{-\frac{1}{2(1-\theta_3^2)^{1/2}\theta_2}\left\{\frac{X_1^2}{\theta_1}+\theta_1X_2^2-2\theta_3X_1X_2\right\}\right\}. \tag{2}$$

With this reparameterization, the Fisher Information matrix reduces to

$$\mathbf{I}(\theta_1,\theta_2,\theta_3)=\mathrm{Diag}(\theta_1^{-2}(1-\theta_3^2)^{-1},\theta_2^{-2},(1-\theta_3^2)^{-2}). \tag{3}$$

This establishes immediately the mutual orthogonality of θ_1, θ_2 and θ_3 in the sense of Huzurbazar (1950) and Cox and Reid (1987). Such orthogonality is often referred to as "Fisher Orthogonality". For subsequent sections, we also need a few other results which are collected in the following lemma.

Lemma 1. *For the bivariate normal density given in* (2),

$$\mathbf{E}\left(\frac{\partial^3\log f}{\partial\theta_3^2\,\partial\theta_1}\right)=0, \quad \mathbf{E}\left(\frac{\partial^3\log f}{\partial\theta_3^2\,\partial\theta_2}\right)=\frac{1}{\theta_2(1-\theta_3^2)^2}, \quad \mathbf{E}\left(\frac{\partial^3\log f}{\partial\theta_3^3}\right)=-\frac{6\theta_3}{(1-\theta_3^2)^3}; \tag{4}$$

$$\mathbf{E}\left(\frac{\partial\log f}{\partial\theta_3}\times\frac{\partial^2\log f}{\partial\theta_3^2}\right)=\frac{2\theta_3}{(1-\theta_3^2)^3}, \quad \mathbf{E}\left(\frac{\partial^3\log f}{\partial\theta_3\,\partial\theta_1^2}\right)=-\frac{\theta_3}{\theta_1^2(1-\theta_3^2)^2}, \quad \mathbf{E}\left(\frac{\partial^3\log f}{\partial\theta_3\,\partial\theta_2^2}\right)=0; \tag{5}$$

We derive the matching priors in the next few sections.

3. Quantile matching priors

The main objective of quantile matching priors is to construct one-sided credible intervals for real-valued parameters based on posterior quantiles, the coverage probabilities of which matches asymptotically the corresponding frequentist