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Optimal detection of Fechner-asymmetry

Delphine Cassart[a], Marc Hallin[a, b, *, 1], Davy Paindaveine[a, b, 2]

[a] Institut de Recherche en Statistique, E.C.A.R.E.S., Université Libre de Bruxelles, Brussels, Belgium
[b] Département de Mathématique, Université Libre de Bruxelles, Brussels, Belgium

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Abstract

We consider a general class of skewed univariate densities introduced by Fechner [1897. Kollectivmasslehre. Engleman, Leipzig], and derive optimal testing procedures for the null hypothesis of symmetry within that class. Locally and asymptotically optimal (in the Le Cam sense) tests are obtained, both for the case of symmetry with respect to a specified location as for the case of symmetry with respect to some unspecified location. Signed-rank based versions of these tests are also provided. The efficiency properties of the proposed procedures are investigated by a derivation of their asymptotic relative efficiencies with respect to the corresponding Gaussian parametric tests based on the traditional Pearson–Fisher coefficient of skewness. Small-sample performances under several types of asymmetry are investigated via simulations.

MSC: 62M15; 62G35

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1. Introduction

1.1. Testing for symmetry

Symmetry is one of the most important and fundamental structural assumptions in statistics, playing a major role, for instance, in the identifiability of location or intercept under nonparametric conditions: see for instance Stein (1956), Beran (1974), and Stone (1975). This importance explains the huge variety of existing testing procedures of the null hypothesis of symmetry in an i.i.d. sample $X_1, \ldots, X_n$.

Classical tests of the null hypothesis of symmetry—the hypothesis under which $X_1 - \theta \stackrel{\mathrm{d}}{=} -(X_1 - \theta)$ for some location $\theta \in \mathbb{R}$, where $\stackrel{\mathrm{d}}{=}$ stands for equality in distribution—are based on third-order moments. Let $m_k^{(n)}(\theta) := n^{-1}\sum_{i=1}^n (X_i - \theta)^k$ and $m_k^{(n)} := m_k^{(n)}(\bar{X}^{(n)})$, where $\bar{X}^{(n)} := n^{-1}\sum_{i=1}^n X_i$. When the location θ is specified, the test

* Corresponding author at: Institut de Recherche en Statistique, E.C.A.R.E.S., Université Libre de Bruxelles, Brussels, Belgium.
E-mail address: mhallin@ulb.ac.be (M. Hallin).

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statistic is

$$n^{1/2}m_3^{(n)}(\theta)/(m_6^{(n)}(\theta))^{1/2}, \tag{1.1}$$

the null distribution of which, under finite sixth-order moments, is asymptotically standard normal. When θ is unspecified, the classical test is based on the empirical coefficient of skewness $b_1^{(n)} := m_3^{(n)}/s_n^3$, where $s_n := (m_2^{(n)})^{1/2}$ stands for the empirical standard error in a sample of size n. More precisely, this test relies on the asymptotic standard normal distribution (still under finite moments of order 6) of

$$n^{1/2}m_3^{(n)}/(m_6^{(n)} - 6s_n^2 m_4^{(n)} + 9s_n^6)^{1/2}.$$

These tests are generally considered as Gaussian procedures, although they cannot be considered optimal in any Gaussian sense, as asymmetric alternatives clearly do not belong to the Gaussian world.

Now, symmetry typically is a nonparametric hypothesis. Nonparametric tests of symmetry based on empirical distribution functions have been proposed by Butler (1969) (with a test statistic of the Kolmogorov–Smirnov type), Rothman and Woodrofe (1972), and Hill and Rao (1977) (with a test statistic of the Cramér–von Mises type)—to quote only a few of them. As usual in such context, some arbitrary distance is adopted on the space of distribution functions, but no optimality issues are addressed; consistency rates are nonparametric. The null hypothesis of symmetry on the other hand also enjoys a rich group invariance structure, that should not remain unexploited. Maximal invariance arguments in this context naturally bring *signs* and *signed ranks* into the picture. The most popular nonparametric signed-rank tests of symmetry (with respect to a specified location θ) are the *sign test*, based on the binomial distribution of the number of negative signs (of $X_i - \theta$) in a sample of size n, and *Wilcoxon's signed-rank test*, based on the exact or asymptotic null distribution of $S_W^{(n)} := n^{-1/2}\sum_{i=1}^n s_i R_{+,i}^{(n)}$, where $s_1, \ldots, s_n$ denote the signs, and $R_{+,1}^{(n)}, \ldots, R_{+,n}^{(n)}$ the ranks of the absolute values, of the deviation $X_i - \theta$ in a sample of size n. Other signed-rank tests of symmetry have been proposed by McWilliams (1990) and Randles et al. (1980)—where the ranks and signs however are not those of the observations themselves, and are not distribution-free under the null hypothesis.

Again, these tests are not optimal in any satisfactory sense against asymmetry. The Wilcoxon and sign tests actually are locally asymptotically optimal against location shifts, under logistic or double-exponential densities, respectively. The sign test is even completely insensitive to asymmetric alternatives which preserve the median, i.e., do not include any "shift component".

The main objective of this paper is to provide, for the problem of testing for symmetry (that is, $f(x-\theta)=f(\theta-x)$ for all x, with specified or unspecified location θ), a concept of optimality that coincides with practitioners' intuition, and to construct parametric and signed-rank tests achieving such optimality. This requires embedding the null hypothesis of symmetry into adequate families of asymmetric alternatives. One of the simplest families of asymmetric densities has been proposed more than a century ago by Fechner (1897), and was recently revived by Arellano-Valle et al. (2005). Denote by f_1 a symmetric (with respect to the origin) standardized density. The f_1-*Fechner family* is the three-parameter collection of densities of the form

$$f_{\theta,\sigma,\xi}(x) := \frac{1}{\sigma}\left[f_1\left(\frac{x-\theta}{(1+\xi)\sigma}\right)I[x \leqslant \theta] + f_1\left(\frac{x-\theta}{(1-\xi)\sigma}\right)I[x > \theta]\right] = \frac{1}{\sigma}f_1\left(\frac{x-\theta}{\sigma(1-\xi\,\mathrm{sign}(x-\theta))}\right), \tag{1.2}$$

$x \in \mathbb{R}$, indexed by location ($\theta \in \mathbb{R}$), scale ($\sigma \in \mathbb{R}_0^+$), and a *skewness parameter* $\xi \in (-1, 1)$. Clearly, $\xi = 0$, in such families, indicates symmetry, $\xi > 0$ asymmetry to the left, and $\xi < 0$ asymmetry to the right. Fechner families thus provide a simple and easily interpretable model for location, scale, and skewness.

The families considered by Arellano-Valle et al. (2005) are more general, with densities of the form

$$\frac{2}{(a(\xi)+b(\xi))\sigma}\left[f\left(\frac{x-\theta}{b(\xi)\sigma}\right)I[x \leqslant \theta] + f\left(\frac{x-\theta}{a(\xi)\sigma}\right)I[x > \theta]\right], \tag{1.3}$$

where $\xi \mapsto a(\xi)$ is monotone increasing and $\xi \mapsto b(\xi)$ monotone decreasing, with $a(0)=1=b(0)$, so that $\xi=0$ yields symmetry, $\xi > 0$ asymmetry to the left, and $\xi < 0$ asymmetry to the right. No obvious choice exists for the functions $a(\xi)$ and $b(\xi)$, and Fechner families clearly correspond to the simple "linear" case $a(\xi)=1-\xi$ and $b(\xi)=1+\xi$. Now, if $a(\xi)$ and $b(\xi)$ are smooth enough in a neighborhood of $\xi=0$, they are of the form

$$a(\xi) = 1 + a\xi + \mathrm{o}(\xi^2) \quad \text{and} \quad b(\xi) = 1 - b\xi + \mathrm{o}(\xi^2), \tag{1.4}$$