Available online at www.sciencedirect.com
ScienceDirect
Journal of Statistical Planning and Inference 137 (2007) 3753–3766
journal of statistical planning and inference
www.elsevier.com/locate/jspi

P^3 approach to intersection–union testing of hypotheses[☆]

Ashis SenGupta[a, b, *]

[a] *Applied Statistics Unit, Indian Statistical Institute, Kolkata, India*
[b] *Department of Statistics, University of California, Riverside, CA, USA*

Available online 27 April 2007

Abstract

Recent applications of statistics often lead one to encounter testing problems where the original hypothesis of interest comprises the union of several sub-hypothesis. In the framework of such intersection–union (IU) testing of hypothesis, in contrast to the usual union–intersection (UI) framework, a sub-hypothesis therein may specify a parameter or a function of some of the parameters of the underlying distribution. The parameters may even be constrained to lie on the boundary of their parameter spaces. Even large-sample tests such as the usual likelihood ratio. Lagrangian multiplier or the Wald's tests then do not apply as their usual asymptotic distribution theory remain no longer be valid. An approach based on a pivotal parametric product P^3 is enhanced here. It is shown that this approach often leads to appealing simple and elegant test statistics. The exact cut-off points and the power values can be computed by judicious use of numerical packages. L-optimality of such a test for the mixture problem is established. For multivariate multiparameter testing problems it is shown that such an approach leads to UI–IU tests. Construction of such tests are exemplified through several real-life problems as in, e.g., testing for interval specifications in acceptance sampling, for generalized variance of structured correlation matrices in generalized canonical variable, for agreement in method comparison studies, for no contamination in multiparameter multivariate mixture models, etc. It is demonstrated for a real-life data set in an acceptance sampling problem that the proposed class of P^3 tests includes the intuitive one existing in the literature.

MSC: primary 62H15 secondary 62P10 62P30

Keywords: Acceptance sampling; Exact optimal tests; Multivariate mixture models; Pivotal parametric product; Union–intersection intersection–union tests

1. Introduction

S.N. Roy's principle of construction of tests for the case when the null hypothesis H_0 consists of the simultaneous occurrence of several disjoint sub-hypotheses and is represented as $\mathrm{H}_0 = \bigcap_{i=1}^{s} \mathrm{H}_{0i}$ is well known as the union–intersection (UI) principle (Roy, 1953). The reverse scenario, i.e., where H_0 holds when at least any one of H_{0i} holds, i.e., $\mathrm{H}_0 = \bigcup_{i=1}^{s} \mathrm{H}_{0i}$, referred to as the intersection–union (IU) testing of hypotheses problem in SenGupta (1991), is also faced in practice and is recently attracting quite some attention. While the celebrated UI testing procedure of Roy is mainly enhanced for multiparameter testing problems in multivariate distributions, the IU testing problems

[☆] Invited paper prepared for the S.N. Roy centenary volume of *Journal of Statistical Planning and Inference*.

[*] Applied Statistics Unit, Indian Statistical Institute, Kolkata, India. Tel.: +91 33 2575 2800.
E-mail address: ashis@isical.ac.in.

0378-3758/$ - see front matter © 2007 Published by Elsevier B.V.
doi:10.1016/j.jspi.2007.03.047

arise in important one-parameter situations also, e.g., in some recently emerging areas such as bioequivalence or generally "equivalence" testing problems (see, e.g., Choudhary and Nagaraja, 2004; Mandallaz and Mau, 1981), acceptance sampling in statistical process control (SPC) (Berger, 1982), reliability and multivariate analysis. This problem also arises of course for multiparameter problems, of which special mention must be made of the test for no mixture in contaminated or mixture models. Sometimes the standard separate tests for each H_{0i} may be combined, to yield a test for H_0 as, say, with the critical region (c.r.) given by the intersection of the separate c.r.s, see, e.g., Choudhary and Nagaraja (2004). However, the determination of the cut-off points there seems as to be often done in an intuitive manner. Also, as, e.g., is exemplified below by the mixture models, it is not always even possible to have exact 'separate' tests. This is so because the elimination of even location or scale, and of course a non-location–scale, nuisance parameter poses non-trivial problems. A unified approach motivated by optimality considerations and based on pivotal parametric product (P^3) (SenGupta, 1991) and its unbiased estimating function is pursued here and is shown to yield simple and elegant *exact* tests for a variety of situations including those mentioned above. An application of the P^3 test to yield a useful test that can be implemented in practice in lieu of the trivial UMP test (Lehmann, 1986, Additional Problems 53, p. 126) is also presented here. Examples also include tests for no contamination for linear random variables and for isotropy for circular random variables. Determination of exact cut-off points and derivation of exact power are also illustrated through an important real-life problem from acceptance sampling. Further, the fact that the class of P^3 tests can have attractive power performance is exemplified by the superiority of such a test over some ad-hoc ones and also by demonstrating that it includes the existing intuitive one for this problem through exact power computations. Additionally, such P^3 tests are shown to be capable of yielding even UMP and L-optimal tests. Finally, it is demonstrated interestingly that for the general situation of multivariate multiparameter IU testing problems, application of Roy's UI principle on the optimal P^3 tests is a powerful method that can yield elegant tests. Such tests are named here as the union-intersection–intersection-union tests.

2. Definition and construction of a P^3 test

We note that for the IU testing problem, the likelihood ratio tests (LRTs) can be quite cumbersome, e.g., when some of the H_{0i}s constrain the parameter(s) in certain intervals. The LRTs may even lack their usual asymptotic properties, e.g., when one of the H_{0i}s, as written above, specifies a parameter value lying on the boundary of the parameter space. This is very common, e.g., in the framework of testing for no contamination in mixture models as will be taken up later. A simpler procedure which is based on an optimality approach is proposed here. The idea is to first recast the original multiple H_{0i} representation of the null hypothesis in terms of only a single hypothesis involving an appropriately chosen parametric function. We will call such a parametric function a *pivotal parametric product* (P^3 in short).

Definition 1. A scalar parametric function η of a possibly vector-valued parameter θ, i.e., $\eta \equiv g(\theta)$, will be termed a pivotal parametric product, P^3 in short, when the null hypothesis $H_0 \equiv \bigcup_{i=1}^{s} H_{0i}$ for θ holds if and only if $\eta = 0$.

After an appropriate P^3 is chosen, the test for $H_0 : \eta = 0$ will be constructed on the basis of a suitable (usually unbiased and/or consistent) estimator, say $\hat{\eta}$, of it. The c.r. will be defined based on how the original alternative translates, one-sided or two-sided, in terms of η. We will call such a test a P^3 test. In general an IU testing problem will not admit of the existence of the UMP test. Attention will therefore be then focused to locally optimal procedures, e.g., 'L-optimal' tests introduced by SenGupta (1991) and further pursued by Pal and SenGupta (2000) in the context of mixture problems.

3. Case of one-parameter H_0

Let a random variable X follow the distribution $f(x; \theta)$, where θ is a scalar parameter. Here we consider testing $H_0 : \theta \leqslant \theta_1$ or $\theta \geqslant \theta_2$ vs $H_1 : \theta_1 < \theta < \theta_2$.

H_0 can be represented as $\bigcap_{i=1}^{2} H_{0i}$ where the sub-hypotheses H_{01} and H_{02} are given by, $H_{01} : \theta \leqslant \theta_1$ and $H_{02} : \theta \geqslant \theta_2$.

A P^3 can be easily identified as $\eta = (\theta - \theta_1)(\theta - \theta_2)$ and H_0 and H_1 translate in terms of η to $H_0 : \eta \geqslant 0$ and $H_1 : \eta < 0$, respectively. A c.r. of the test based on η can then be suggested as $\omega : \hat{\eta} \leqslant K$, where $\hat{\eta}$ is an unbiased estimator of η and K is a constant to be determined such as to meet the desired size α. However, noting that η is a quadratic function of θ, this test can be equivalently represented by the test $\phi(\hat{\theta})$ defined by the c.r. (assuming the non-randomized setup