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On a better lower bound for the frequentist probability of coverage of Bayesian credible intervals in restricted parameter spaces

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ABSTRACT

For estimating a lower restricted parametric function in the framework of Marchand and Strawderman (2006), we show how $(1-\alpha) \times 100\%$ Bayesian credible intervals can be constructed so that the frequentist probability of coverage is no less than $1-\frac{3\alpha}{2}$. As in Marchand and Strawderman (2013), the findings are achieved through the specification of the *spending function* of the Bayes credible interval and apply to an "equal-tails" modification of the HPD procedure among others. Our results require a logconcave assumption for the distribution of a pivot, and apply to estimating a lower bounded normal mean with known variance, and to further examples include lower bounded scale parameters from Gamma, Weibull, and Fisher distributions, with the latter also applicable to random effects analysis of variance.

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1. Introduction

1.1. Matching frequentist probability of coverage and Bayesian credibility in unrestricted parameter space problems

Frequentist coverage probability is an interesting and informative measure of the efficiency of a Bayes credible set procedure, in particular when the latter is generated through a default or non-informative prior. Of course, it has long been known that there are certain situations (e.g., [6]) where a Bayes $1-\alpha$ credible set can be chosen to have exact probability coverage $1-\alpha$. Examples of such procedures include basic tools in the statistician's arsenal such as the z and t intervals $\bar{x} \pm z_{\alpha/2}\sigma/\sqrt{n}$ and $\bar{x} \pm t_{\alpha/2}s/\sqrt{n}$ with exact frequentist coverage probability and exact Bayes credibility $1-\alpha$, arising for samples from a $N(\mu, \sigma^2)$ population, and the non-informative priors $\pi(\mu) = 1$ and $\pi(\mu, \sigma) = \frac{1}{\sigma}$ respectively. There are a vast class of location, scale, or location-scale family inference problems (e.g., see [8,9], for examples) where there is a match between the credibility and frequentist probability coverage of Bayes confidence intervals, and which relate to the contributions of this paper.

Example 1. Consider an observable X with Lebesgue density $f(x;\theta)$, $x \in \mathcal{X}$, $\theta \in \Theta \subset \mathbb{R}^p$ and the problem of estimating a parametric function $\tau(\theta)$ ($\mathbb{R}^p \to \mathbb{R}$). Assume there exists a pivot of the form $T(X,\theta) = \frac{a_1(X)-\tau(\theta)}{a_2(X)}$; $a_2(\cdot) > 0$; such that $-T(X,\theta)$ has cdf G and Lebesgue density g. Observe at this point that if c, d are such that $G(d) - G(c) = 1-\alpha$, then the confidence interval

$$I(X) = [\, a_1(X) + ca_2(X), a_1(X) + da_2(X)\,] \tag{1}$$

has frequentist probability of coverage $P_\theta(I(X) \ni \tau(\theta)) = 1-\alpha$, for all $\theta \in \Theta$.

Further assume that the family of densities for X is invariant under a group $\mathcal{G}$ of transformations and that the pivot satisfies the invariance requirement $T(x,\theta) = T(gx, \bar{g}\theta)$, for all $x \in \mathcal{X}$, $\theta \in \Theta$, $g \in \mathcal{G}$, $\bar{g} \in \bar{\mathcal{G}}$, with $\mathcal{X}$, Θ, G, and $\bar{G}$ being isomorphic. Relative to this group structure, consider the Haar right invariant prior π_H.[1] A key feature of this choice of prior is that

$$T(x,\theta)|x =^d T(X,\theta)|\theta, \quad \text{for all } x, \theta, \text{ when } \theta \sim \pi_H, \tag{2}$$

in other words the posterior distribution of $-T(x,\theta)$ is free of x and matches the pivotal distribution of $-T(X,\theta) \sim G$ (see [8], Corollary 1, for more details).

Now given property (2), the confidence interval in (1) has credibility

$$P(\tau(\theta) \in I(x)|x) = P(-T(x,\theta) \in [c,d]|x) = 1-\alpha, \quad \text{for all } x,$$

which matches indeed the frequentist probability of coverage.

1.2. Unmatching and challenges in the presence of parametric restrictions

Now, consider the context of Example 1, but with the parametric restriction $\tau(\theta) \ge a$ for some known a. Such a restriction arises naturally in many settings, such as elicited in a problem of estimating the mass of a neutrino (e.g., [7,4]) and in random effects analysis of variance (see Example 3). Clearly, the truncation of $I(X) \cap [a, \infty)$ preserves frequentist probability of coverage $1-\alpha$ for the restricted parameter space $\{\theta : \tau(\theta) \ge a\}$, but it is not a Bayes credible set anymore. As discussed by Mandelkern [7], for estimating the mean μ of a $N(\mu, \sigma^2)$ distribution with known σ^2, several frequentist based and Bayesian options remain but they differ. Namely, the $(1-\alpha) \times 100\%$ highest posterior density (HPD) Bayes credible set associated with the prior $\pi_0(\theta) = \pi_H(\theta)\,\mathbb{I}_{[a,\infty)}(\tau(\theta))$, i.e. the truncation of π_H on the restricted parameter space, has frequentist probability of coverage

[1] This satisfies the property $\pi_H(A\bar{g}) = \pi_H(A)$ for every measurable subset A of Θ, and for every $g \in G$. Such a measure π_H exists and is unique up to a multiplicative constant for locally compact groups such as location, scale, and location-scale. We refer to Berger [2] or Eaton [3] for detailed treatments of invariance and Haar invariant measures.

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