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Improved estimators for parameters of a Pareto distribution with a restricted scale

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ABSTRACT

We consider the problem of estimating the parameters of a Pareto distribution under a quadratic loss when the scale parameter is constrained. The integral expression of risk difference (IERD), the approach of Kubokawa (1994) [12], and the Brewster and Zidek's (1974) [5] technique are used to obtain the improved estimators over the standard estimators. Some complete class results are also proved.

1. Introduction

Estimation problems under restricted parameter spaces have received considerable attention for the past sixty years. Investigations in several physical, biological, clinical and socio-economic experiments lead to a priori information about unknown parameters of the concerned populations. The rate of cure of a newly introduced drug is likely to be more than the one of its earlier version. The average yield of a crop using fertilizers is expected to be more than the one not using any fertilizers. The incorporation of a priori knowledge about unknown parameters in an estimation problem leads to better (more efficient) procedures. One may refer to Moors [16,17], Kumar and Sharma [14], Bader and Bischoff [3], Kubokawa [13] and van Eeden [25] for a detailed review on estimation under restricted parameter spaces.

In this article, we consider the problem of estimating unknown parameters associated with a Pareto $P(\alpha, \beta)$ distribution under restrictions on the parameter α. The probability density function of a Pareto

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$P(\alpha, \beta)$ distribution is given by

$$f(x, \alpha, \beta) = \frac{\beta\, \alpha^{\beta}}{x^{\beta+1}}, \quad \alpha \le x < \infty,\ 0 < \alpha < \infty,\ \beta > 0. \tag{1.1}$$

Here α and β are scale and shape parameters respectively. In particular, α represents the minimum wage while describing income distributions. In this paper, the estimation of α and β is considered under the assumption that α is bounded above by a known constant $\alpha_0(> 0)$. Without loss of generality, α_0 is taken as 1. The distribution $P(\alpha, \beta)$ was first introduced by Pareto [18] as a model for income data. Thereafter this distribution has found applications in many agricultural, industrial and economic investigations, see for example, [9,15,10]. For a review of the work on the problem of estimating parameters of a Pareto population, one may refer to Asrabadi [2], Saksena and Johnson [20], Baxter [4], Kern [11], Rytgaard [19], Aban, Meerschaert and Panorska [1], Dixit and Nooghabi [6,7], van Zyl [26] and Scollnik [21].

The estimation problems in Pareto distribution when one of the parameters may be bounded arises in practice. Dixit and Nooghabi [7] consider a real data set on motor insurance claims to follow a Pareto distribution, where α is the amount below which claims are not entertained. Now due to policies of the insurance company, α may not be allowed to be higher than a prescribed value α_0, which is determined by the amount of premium charged from the customer.

The rest of the article is organized as follows. In Section 2, estimation of α, when it is restricted to $0 < \alpha \le 1$ with β known, is considered. A complete class of estimators is derived in Section 2.1 and it is shown to contain the restricted maximum likelihood estimator (MLE) and the Bayes estimator with respect to the uniform prior. Further, a class of estimators improving upon the best scale equivariant (BSE) estimator is obtained in Section 2.2. The Bayes estimator with respect to the uniform prior over the restricted parameter space is shown to belong to this class. In Section 3, estimation of β is considered with unknown α, $0 < \alpha \le 1$. Further in Section 3.1, a complete class result is proved and as a consequence, the restricted MLE has been shown to be inadmissible. A class of dominating estimators is also derived. In Section 3.2, a class of estimators improving upon the best affine equivariant (BAE) estimator is obtained using the IERD approach of Kubokawa [12]. Furthermore, a complete class result is established for scale equivariant estimators in Section 3.3. A Stein-type estimator is considered which belongs to these classes. The risk functions of improving estimators are compared numerically. As an illustration of the methods in this section a real data set on Pareto distribution is considered.

2. Estimation of the scale parameter when the shape parameter is known

Suppose $X_1, X_2, \ldots, X_n$ is a random sample taken from a Pareto $P(\alpha, \beta)$ distribution with $0 < \alpha \le 1$ and known β. A complete and sufficient statistic is $X_{(1)}$ having a $P(\alpha, n\beta)$ distribution where $X_{(1)} = \min(X_1, X_2 \ldots, X_n)$ (see, [20]). We will use the notation $X = X_{(1)}$ in the rest of this section. For further considerations, it is assumed that $n\beta > 2$. We estimate α with respect to a scale invariant quadratic loss function defined by

$$L(\alpha, d) = \left(\frac{d}{\alpha} - 1\right)^2. \tag{2.1}$$

Note that the problem of estimating α under the loss function (2.1), with no restrictions on the parameters space, is invariant under the scale group of transformations. The form of a scale equivariant estimator is easily seen to be $\hat{\alpha}_a = ax$. Minimizing the risk of $\hat{\alpha}_a$ with respect to a, we get the BSE estimator of α as

$$\delta_{BS}(X) = a_0 X \tag{2.2}$$

where $a_0 = \frac{n\beta-2}{n\beta-1}$.

2.1. A complete class result

Maximizing the likelihood function over the restricted parameter space $0 < \alpha \le 1$, we find that the MLE of α is $\min(X, 1)$. In this subsection, we seek improving estimators in a class of estimators