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The total time on test transform and the decreasing percentile residual life aging notion

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ABSTRACT

Recently Nair and Sankaran (2013) listed some known characterizations of common aging notions in terms of the total time on test transform (TTT) function. They also derived some new characterizations. The purpose of this note is to add two characterizations of the decreasing percentile residual life of order α (DPRL(α)) aging notion in terms of the TTT function, and in terms of the observed TTT when X is observed.

1. Introduction

Let F be a distribution function of a nonnegative random variable X, and let $Q = F^{-1}$ be the corresponding (left-continuous) *quantile function*, defined by

$$Q(u) = \inf\{x \big| F(x) \geq u\}, \quad 0 \leq u \leq 1.$$

Denote the survival function of X by $\overline{F} \equiv 1 - F$. The *total time on test transform* (TTT) *function*, T, is defined as

$$T(u) = \int_0^{Q(u)} \overline{F}(x)\, dx, \quad 0 \leq u \leq 1. \tag{1.1}$$

Note that $T(1) = E[X]$, where the expectation $E[X]$ can be finite or infinite.

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The concept of the TTT transform is useful in reliability theory. A recent paper that describes some of its applications, and that contains a list of some basic references, is the paper by Nair and Sankaran [11]. An application of the TTT transform in actuarial science is described in Li and Shaked [9, Remark 2.4].

Let u_X be the right endpoint of the support of X; u_X may be finite or infinite. For any $s < u_X$, the *residual life* at time s, that is associated with X, is any random variable that has the conditional distribution of $X - s$ given that $X > s$. We denote it by

$$X_s = [X - s \big| X > s], \quad s \in [0, u_X).$$

The *α-percentile residual life function* P_α, where α is some number between 0 and 1, is defined by

$$P_\alpha(s) = \begin{cases} F_{X_s}^{-1}(\alpha), & t \in [0, u_X); \\ 0, & s \geq u_X. \end{cases}$$

It is useful to note (see Franco-Pereira, Lillo, and Shaked [3]) that

$$P_\alpha(s) = Q(\alpha + (1-\alpha)F(s)) - s, \quad s \in [0, u_X). \tag{1.2}$$

For $\alpha \in (0, 1)$, the random variable X is said to be DPRL(α) if

$$P_\alpha(s) \text{ is decreasing in } s \in (0, \infty) \tag{1.3}$$

(here, and in the rest of this paper, "decreasing" and "increasing" are used in the non-strict sense). A basic paper on the DPRL(α) notion is Haines and Singpurwalla [5]. The DPRL(α) aging notion was further studied in more depth in Franco-Pereira, Lillo, and Shaked [3] and in references therein.

Recently Nair and Sankaran [11] listed some known characterizations of common aging notions in terms of the TTT function. They also derived some new characterizations. The purpose of this note is to add two characterizations of the DPRL(α) aging notion in terms of the TTT function, and in terms of the *observed* TTT *when X is observed* (the latter will be formally defined in Section 3).

2. A characterization in terms of the TTT density

In this section we assume that Q is differentiable almost everywhere on $[0, 1)$; that is, that F has at most a countable number of "flats".

The *quantile density* q of X is defined, almost everywhere on $[0, 1)$, by

$$q(u) = \frac{d}{du}Q(u).$$

The TTT *density* t of X is defined, almost everywhere on $[0, 1)$, by

$$t(u) = \frac{d}{du}T(u). \tag{2.1}$$

An expression of the quantile function Q in terms of the TTT density t (that will be used in the sequel) is given in the following proposition. The relation (2.3) in the following proposition is a special case of a result in Nair, Sankaran, and Vineshkumar [12, page 1130], whereas the relation (2.2) can be easily derived from (2.3).

Proposition 2.1. *Suppose that Q is differentiable almost everywhere on* $[0, 1)$. *Then*

$$Q(u) = \int_0^u \frac{t(v)}{1-v}\, dv \tag{2.2}$$

almost everywhere on $[0, 1)$*, and*

$$q(u) = \frac{t(u)}{1-u} \tag{2.3}$$

almost everywhere on $[0, 1)$.