Statistical Methodology 18 (2014) 41–63

Contents lists available at ScienceDirect

Statistical Methodology

journal homepage: www.elsevier.com/locate/stamet

Selecting the best of two gamma populations having unequal shape parameters

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ARTICLE INFO

Article history:
Received 21 March 2013
Received in revised form
22 August 2013
Accepted 25 August 2013

Keywords:
Admissible
Best population
Correct selection
Generalized Bayes
Indifference-zone
Minimax selection rule
Preference-zone

ABSTRACT

Consider two independent gamma populations π_1 and π_2, where the population π_i has an unknown scale parameter $\theta_i > 0$ and known shape parameter $\alpha_i > 0, i = 1, 2$. Assume that the correct ordering between θ_1 and θ_2 is not known a priori and let $\theta_{[1]} \leq \theta_{[2]}$ denote the ordered values of θ_1 and θ_2. Consider the goal of identifying (or selecting) the population associated with $\theta_{[2]}$, under the indifference-zone approach of Bechhofer (1954), when the quality of a selection rule is assessed in terms of the infimum of the probability of correct selection over the preference-zone. Under the decision-theoretic framework this goal is equivalent to that of finding the minimax selection rule when (θ_1, θ_2) lies in the preference-zone and 0–1 loss function is used (which takes the value 0 if correct selection is made and takes the value 1 if correct selection is not made). Based on independent observations from the two populations, the minimax selection rule is derived. This minimax selection rule is shown to be generalized Bayes and admissible. Finally, using a numerical study, it is shown that the minimax selection rule outperforms various natural selection rules.

1. Introduction

Gamma distribution is widely used in survival analysis, reliability engineering and life testing to provide representations of various physical situations. Consider two different gamma distributions representing a characteristic of two different physical situations (say, lifetimes of a product manufactured using two different manufacturing processes). Let π_1 and π_2 denote the populations comprising of realizations of the characteristic of two physical situations (say, populations of lifetimes

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1572-3127/$ – see front matter
http://dx.doi.org/10.1016/j.stamet.2013.08.008

of units manufactured using the two manufacturing processes). It may be of interest to select the best of two populations, where the quality of a population is assessed in terms of a function of parameters associated with the probability distribution representing it (say, the quality of a manufacturing process is assessed in terms of the mean lifetime of the units produced through it). To make the above discussion precise, let π_1 and π_2 be two independent gamma populations such that the observation X_i from the population π_i has the Lebesgue probability density function (p.d.f.)

$$f_i(x|\theta_i) = \begin{cases} \dfrac{1}{\Gamma(\alpha_i)\theta_i^{\alpha_i}} x^{\alpha_i - 1} e^{-\frac{x}{\theta_i}}, & \text{if } x > 0 \\ 0, & \text{otherwise}, \end{cases} \qquad i = 1, 2,$$

where $\Gamma(\alpha)$, $\alpha > 0$, is the usual gamma function, $\alpha_1 > 0$ and $\alpha_2 > 0$ are known shape parameters, and $\theta_1 > 0$ and $\theta_2 > 0$ are unknown scale parameters. Note that if $X_{i1}, \ldots, X_{in_i}$ is a random sample of size n_i from the population π_i, then $T_i = \sum_{j=1}^{n_i} X_{ij}$ is a complete and sufficient (and hence minimal sufficient) statistic for $\{f(\cdot|\theta_i) : \theta_i > 0\}$, $i = 1, 2$. Moreover T_i again has gamma distribution with the same scale parameter θ_i and known shape parameter $\beta_i = n_i\alpha_i$, $i = 1, 2$. Thus, without loss of generality, we take $n_1 = n_2 = 1$. Let $\Omega = \{(\theta_1, \theta_2) : \theta_i > 0, i = 1, 2\}$ denote the parameter space and let $\chi = \{(x_1, x_2) : x_i > 0, i = 1, 2\}$ denote the sample space. Let $\theta_{[1]} = \min(\theta_1, \theta_2)$ and $\theta_{[2]} = \max(\theta_1, \theta_2)$ so that $\theta_{[1]} \le \theta_{[2]}$ are the ordered values of θ_1 and θ_2. Suppose that the quality of the population π_i is assessed in terms of the corresponding scale parameter θ_i, $i = 1, 2$. We say that the population π_1 (π_2) is best if $\theta_1 > \theta_2$ ($\theta_2 > \theta_1$). In case of tie $\theta_1 = \theta_2$, since both the populations are equally good, consequences of making incorrect selection will be null and therefore we may define either of the two populations as the best population. In case of tie $\theta_1 = \theta_2$ we define the population π_2 to be the best population. Thus we say that the population π_1 (π_2) is the best if $\theta_1 > \theta_2$ ($\theta_1 \le \theta_2$). Equivalently we say that the population π_1 (π_2) is worst if $\theta_1 \le \theta_2$ ($\theta_1 > \theta_2$). Let $\pi_{(1)}$ and $\pi_{(2)}$, respectively, denote the worst and the best populations, so that

$$\pi_{(1)} = \begin{cases} \pi_1, & \text{if } \theta_1 \le \theta_2 \\ \pi_2, & \text{if } \theta_1 > \theta_2 \end{cases} \quad \text{and} \quad \pi_{(2)} = \begin{cases} \pi_2, & \text{if } \theta_1 \le \theta_2 \\ \pi_1, & \text{if } \theta_1 > \theta_2. \end{cases}$$

Assume that the scale parameters θ_1 and θ_2 are completely unknown so that the correct pairing between the members of $\{\pi_1, \pi_2\}$ with those of $\{\pi_{(1)}, \pi_{(2)}\}$ is not known a priori. The goal is to identify (or select) the best population $\pi_{(2)}$ based on $\underline{X} = (X_1, X_2)$. A "Correct Selection" (CS) is defined as the event which fulfills this goal, i.e., CS = {population $\pi_{(2)}$ is selected}.

The problem described above may be posed as a statistical decision problem with action space $\mathcal{A} = \{1, 2\}$, where taking action $i \in \mathcal{A}$ corresponds to selection of π_i, $i = 1, 2$, as the best population. A selection rule δ may be defined as a map from χ to [0, 1] such that, for $(x_1, x_2) \in \chi$, $\delta(x_1, x_2)$ is the conditional probability of selecting π_1 as the best population given that $(X_1, X_2) = (x_1, x_2)$, and $1 - \delta(x_1, x_2)$ is the conditional probability of selecting π_2 as the best population given that $(X_1, X_2) = (x_1, x_2)$. Let $\mathcal{D}$ denote the class of all selection rules.

Note that the probability of CS depends on the choice of selection rule and on unknown parameters θ_1 and θ_2. For a given selection rule δ and for given $\underline{\theta} = (\theta_1, \theta_2) \in \Omega$, let $P_{\underline{\theta}}(CS|\delta)$ denote the probability of CS achieved using selection rule δ when $\underline{\theta} \in \Omega$ is the true parametric value. One may like to use a selection rule δ for which

$$P_{\underline{\theta}}(CS|\delta) \ge P^*, \quad \forall\, \underline{\theta} \in \Omega, \quad \text{or equivalently,} \quad \inf_{\underline{\theta} \in \Omega} P_{\underline{\theta}}(CS|\delta) \ge P^*, \tag{1.1}$$

where $P^* \in \left[\frac{1}{2}, 1\right)$ is a pre-assigned real constant. The condition $P^* \ge \frac{1}{2}$ is reasonable in the sense that the probability requirement $P_{\underline{\theta}}(CS|\delta^0) = \frac{1}{2}$ is met by the no-data rule δ^0 $\left(= \frac{1}{2}, \forall\, (x_1, x_2) \in \chi\right)$ which selects one of the two populations at random as the best population. Commonly used values of P^* are 0.9, 0.95 and 0.99. Under quite general conditions it is known that, for any selection rule δ, $\inf_{\underline{\theta} \in \Omega} P_{\underline{\theta}}(CS|\delta) \le \frac{1}{2}$ (cf. Misra and Dhariyal [26]), owing to the fact that θ_1 and θ_2 may be arbitrarily close. Thus the probability requirement (1.1) cannot be met unless some additional conditions are imposed. Based on the observation that if θ_1 and θ_2 are close then the consequences of making the incorrect selection may be negligible, Bechhofer [10] suggested to control the probability of CS only