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Transformations of multivariate Edgeworth type expansions

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ABSTRACT

Given a random vector whose distribution can be expanded in powers of some parameter ε (such as the Edgeworth expansion with $\varepsilon = n^{-1/2}$ and n the sample size), methods are given for expanding the distribution of a transformation of it in powers of ε. Under specified conditions the derived expansion reduces to one in powers of ε^2. Applications are made to lattice and non-lattice random variables and to stationary series.

1. Introduction

The concept of multivariate Edgeworth expansions was introduced by Davis [10]. This concept has attracted applications in many areas of statistics. We mention the following: expansions for Wilks' likelihood ratio statistic [11]; expansions for densities of sufficient estimators [14]; expansions for Hotelling's one-sample T^2 statistic and Roy's largest root statistic [12,13]; expansions for the densities of multivariate M-estimates [17]; expansions for statistics of time series [28]; expansions for the distribution of the standardized squared multiple correlation coefficient [29]; Bartlett-type modification of Rao efficient score statistic [7]; approximation of multivariate densities [21]; expansions for the joint distribution of the sample autocorrelations of a stationary Gaussian long memory process [23]; and, improvement of approximations for the distributions of multinomial goodness-of-fit statistics [25]. The need for multivariate Edgeworth expansions also arise in many applied areas like astronomy, communications, economics, physics and signal processing. See [34,1,2,24,15,16,35].

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Kollo and von Rosen [22] provide a most comprehensive account of the theory and applications of multivariate Edgeworth expansions.

In this paper, we consider problems such as finding the confidence level of an approximate test or confidence region or the power of a test, amounting to finding the distribution of $\mathbf{Y}_n = \mathbf{A}_n(\overline{\mathbf{X}}_1, \ldots, \overline{\mathbf{X}}_k)$, where $\overline{\mathbf{X}}_i$ is the mean of a random sample of n_i transformed observations with expected value $\boldsymbol{\mu}_i$, $\mathbf{A}_n$ is a function having a Taylor series expansion in $n^{-1/2}$, and n is a known parameter such as the minimum or total sample size. We give ways of obtaining asymptotic power series expansions for distributions, such as that of $\mathbf{Y}_n$ when suitably standardized.

When the limiting distribution is normal and $\overline{\mathbf{X}}_1, \ldots, \overline{\mathbf{X}}_k$ are non-lattice, then the Edgeworth expansion provides a solution to this problem in terms of *the cumulant coefficients*—the coefficients in the expansions for the cumulants of $\mathbf{Y}_n$ in powers of n^{-1}. Section 2 deals with the Edgeworth expansion for both normal and non-normal limiting distributions. It is presented in the more natural framework of signed measures rather than being restricted to probability distributions. Included is the expansion for the distribution of $n^{1/2}(\overline{\mathbf{X}} - \boldsymbol{\mu})$ in (a) the lattice case, due to [5], and (b) the case of $\overline{\mathbf{X}}$ the mean of a stationary sequence.

Section 3 shows how to obtain the cumulant coefficients for the limiting normal case. However, difficulties usually arise in the case of non-normal limiting distribution, and an alternative method is given in Section 4. This method gives a formal asymptotic expansion for a transformed distribution (such as that of $\mathbf{Y}_\varepsilon = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{g}_i(\mathbf{X}_\varepsilon)$) when a power series expansion is available for the original distribution (such as that of some random variable $\mathbf{X}_\varepsilon$). In particular, this may be applied to $\mathbf{Y}_\varepsilon$ a suitably standardized version of $\mathbf{Y}_n$ above with $\varepsilon = n^{-1/2}$ and $\mathbf{X}_\varepsilon$ a standardized version of $\overline{\mathbf{X}}_1, \ldots, \overline{\mathbf{X}}_k$, where $\{\overline{\mathbf{X}}_i\}$ may be the means of lattice or stationary sequences.

The following notation is used. Let $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{I}, \mathbb{C}$ denote the non-negative integers, the integers, the reals, the imaginary numbers and the complex numbers. For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^p$ and j in $\mathbb{Z}$, we set

$$|\mathbf{x}|_p = \left(|x_1|, \ldots, |x_p|\right)', \qquad \mathbf{x}! = \prod_{i=1}^{p} x_i!, \qquad \mathbf{xy} = \left(x_1y_1, \ldots, x_py_p\right)',$$

$$\mathbf{x}^j = \left(x_1^j, \ldots, x_p^j\right)', \qquad \mathbf{x}^{\mathbf{y}} = \prod_{i=1}^{p} x_i^{y_i}, \qquad \min(\mathbf{x}, \mathbf{y}) = \left(\min\left(x_1, y_1\right), \ldots, \min\left(x_p, y_p\right)\right)',$$

$$\mathbf{D} = \left(D_1, \ldots, D_p\right)' = \mathbf{D}_{\mathbf{x}} = \partial/\partial\mathbf{x}, \qquad d/d\mathbf{x} = \partial^p/\partial x_1 \cdots \partial x_p, \qquad \dot{f}(\mathbf{x}) = df(\mathbf{x})/d\mathbf{x}.$$

For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^p$, let $\mathbf{x} \le \mathbf{y}$ denote $x_i \le y_i$ for $1 \le i \le p$, let $\mathbf{x} < \mathbf{y}$ denote $\mathbf{x} \le \mathbf{y}$ and $\mathbf{x} \ne \mathbf{y}$, $\mathbf{x} << \mathbf{y}$ denote $x_i < y_i$ for $1 \le i \le p$ and

$$\delta_{\mathbf{xy}} = \begin{cases} 1, & \text{if } \mathbf{x} = \mathbf{y}, \\ 0, & \text{otherwise.} \end{cases}$$

For $\boldsymbol{\nu}$ in $\mathbb{N}^p$ and $\mathbf{g}$ a function on $\mathbb{R}^p$, set

$$|\boldsymbol{\nu}| = \sum_{i=1}^{p} \nu_i, \qquad \mathbf{g}^{(\boldsymbol{\nu})} = \mathbf{D}^{\boldsymbol{\nu}}\mathbf{g} = \prod_{i=1}^{p} D_i^{\nu_i}\mathbf{g}$$

and let $\boldsymbol{\alpha}_{\boldsymbol{\nu}}$ be the $|\boldsymbol{\nu}|$-vector $(\mathbf{1}^{\nu_1}, \ldots, \mathbf{p}^{\nu_p})'$, where $\mathbf{p}^j$ is the j-vector $(p, \ldots, p)'$.

For $1 \le i \le p$, let $\mathbf{e}_{ip} = (\mathbf{0}^{i-1}, 1, \mathbf{0}^{p-i})'$, the ith unit vector in $\mathbb{R}^p$. For $\boldsymbol{\alpha}$ in $\mathbb{N}^r$, let $\boldsymbol{\nu}_{\boldsymbol{\alpha}} = \sum_{i=1}^{r} \mathbf{e}_{\alpha_i p}$, the inverse of $\boldsymbol{\alpha}_{\boldsymbol{\nu}}$ for $\alpha_1 \le \alpha_2 \le \cdots$. In particular, $x_{\alpha_1} \cdots x_{\alpha_r} = \mathbf{x}^{\boldsymbol{\nu}}$ for $\boldsymbol{\nu} = \boldsymbol{\nu}_{\boldsymbol{\alpha}}$ or $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\boldsymbol{\nu}}$ and $r = |\boldsymbol{\nu}|$.

Let $\sum_p = \sum$ and $\sum_p^*$ denote summations over $\mathbb{N}^p$ and $\mathbb{N}^p - \mathbf{0} = \{\boldsymbol{\nu} \text{ in } \mathbb{N}^p \ : \ \boldsymbol{\nu} \ne \mathbf{0}\}$. So, the multinomial expansion for $\mathbf{a} = (a_1, \ldots, a_p)'$ can be written

$$\left[\sum_{i=1}^{p} a_i\right]^k \Big/ k! = \sum_{p, |\boldsymbol{\nu}|=k} \mathbf{a}^{\boldsymbol{\nu}}/\boldsymbol{\nu}!.$$

For a $p \times p$ matrix $\boldsymbol{\Sigma} > \mathbf{0}$ (positive definite), let $\Phi_{\boldsymbol{\Sigma}}, \phi_{\boldsymbol{\Sigma}}$ denote the distribution and density of a $N_p(\mathbf{0}, \boldsymbol{\Sigma})$ random variable, the multivariate normal. If $p = \boldsymbol{\Sigma} = 1$ these are written as Φ, ϕ.