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Confidence distributions: A review[☆]

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ABSTRACT

A review is provided of the concept *confidence distributions*. Material covered include: fundamentals, extensions, applications of confidence distributions and available computer software. We expect that this review could serve as a source of reference and encourage further research with respect to confidence distributions.

Contents

☆ This paper is dedicated to the memory and kindness of Professor Kesar Singh.

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1. Introduction

Point estimators, confidence intervals and p-values have long been fundamental and familiar tools for frequentist statisticians. The notation of *confidence distributions* is less known to statisticians, although it has a long history.

In recent years, there has been a surge of renewed interest in confidence distributions (see, for example, [113,118,119]); so, we feel it is timely to provide a comprehensive review on the subject. In the more recent developments, "the concept of confidence distribution has emerged as a purely frequentist concept. Conceptually, a confidence distribution is no different from a point estimator or an interval estimator (confidence interval), but it uses a sample-dependent distribution function on the parameter space (instead of a point or an interval) to estimate the parameter of interest" [145, p. 8].

Confidence distributions are a way to represent all possible confidence intervals. The area under the confidence density between any two points gives the confidence that the parameter value will lie between those points. Mau [96] stated: "A confidence distribution function is a tool for flexible statistical analyses. It provides one- and two-sided tests of simple and interval hypotheses for any size, and central and symmetric confidence intervals of any level. Given an interval of equivalent values, it quantifies the strength of evidence for 'no material difference' between two populations in a set of data, but is independent of the particular choice of such an interval".

Just as a Bayesian posterior distribution "contains a wealth of information for any type of Bayesian inference, a confidence distribution contains a wealth of information for constructing almost all types of frequentist inferences, including point estimates, confidence intervals and p-values, among others. Some recent developments have highlighted the promising potentials of the confidence distribution concept, as an effective inferential tool" [145, p. 18].

We now give some simple examples of confidence distributions using the concept of pivots. The readers may recall from basic statistics that a pivot is a function of data whose distribution does not depend on population parameters.

For the first example, suppose $x_1, x_2, \ldots, x_n$ is a random sample from a normal distribution with mean μ. Let $\overline{x}$ denote the sample mean and let s denote the sample standard deviation. It is well known $\sqrt{n}\left(\overline{x}-\mu\right)/s$ has the Student's t distribution with $n-1$ degrees of freedom. So, $\sqrt{n}\left(\overline{x}-\mu\right)/s$ is a pivot and a confidence distribution for μ is $\overline{x}+\left(s/\sqrt{n}\right)t_{n-1}=C(\mu)$, say, where t_ν denotes a Student's t random variable with degrees of freedom ν.

For the second example, suppose $x_1, x_2, \ldots, x_n$ is a random sample from a normal distribution with variance σ^2. Let s^2 denote the sample variance. It is well known $(n-1)s^2/\sigma^2$ has the chi-square distribution with $n-1$ degrees of freedom. So, $(n-1)s^2/\sigma^2$ is a pivot and a confidence distribution for σ^2 is $s^2(n-1)/\chi^2_{n-1}=C\left(\sigma^2\right)$, say, where χ^2_ν denotes a chi-square random variable with degrees of freedom ν.

The third example is due to Fraser [42]. Consider a normal distribution with mean μ and variance σ_0^2. The p-value function from some data y is

$$p(\mu)=\int_{-\infty}^{y}\phi\left(\frac{u-\mu}{\sigma_0}\right)du=\Phi\left(\frac{y-\mu}{\sigma_0}\right)$$

which has normal distribution function shape dropping from 1 at $-\infty$ to 0 at $+\infty$ [42]; it records the probability position of the data with respect to a possible parameter value μ [42]. The confidence distribution of μ is normal with mean y and variance σ_0^2. Other details including the corresponding one-sided test can be found in [42].

Confidence distributions are often generated using bootstrap methods and they are a frequentist concept. Several authors (see, for example, [24]) have compared confidence distributions to Bayesian posteriors and give formula for the implied prior if a confidence distribution is used as a posterior.