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Exact distribution of a linear combination of a variable and order statistics from the other two variables of a trivariate elliptical random vector as a mixture of skew-elliptical distributions

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ABSTRACT

We consider here a univariate skew-elliptical distribution, which is a special case of the unified multivariate skew-elliptical distribution studied recently by Arellano-Valle and Azzalini (2006) [1]. We then derive the exact distribution of a linear combination of a variable and order statistics from the other two variables in the case of a trivariate elliptical distribution. We show that the cumulative distribution function (cdf) of this linear combination is a mixture of the univariate skew-elliptical distribution functions.

1. Introduction

A p-dimensional random vector $\mathbf{X}$ is said to have an elliptically contoured distribution with location vector $\boldsymbol{\mu} \in \Re^p$, non-negative definite dispersion matrix $\boldsymbol{\Sigma}$, and characteristic generator φ, if centered random vector $\mathbf{X} - \boldsymbol{\mu}$ has characteristic function of the form $\varphi_{\mathbf{X}-\boldsymbol{\mu}}(\mathbf{t}) = \varphi\left(\mathbf{t}^{\mathrm{T}} \boldsymbol{\Sigma} \boldsymbol{t}\right)$, for $\mathbf{t} \in \Re^p$; see [15] for the most general definition of this family of distributions. In this case, we write $\mathbf{X} \sim EC_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\varphi})$, and denote its cumulative distribution function (cdf) by $F_{EC_p}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\varphi})$. It is well known that this family of distributions is closed under linear transformation, marginalization, and conditioning. In

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particular, if $\mathbf{X} \sim EC_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\varphi})$ with $\boldsymbol{\Sigma}$ of full rank p, then $\boldsymbol{\Sigma}^{-1/2}(\mathbf{X}-\boldsymbol{\mu}) \sim EC_p\left(\mathbf{0}, \mathbf{I}_p, \boldsymbol{\varphi}\right)$, where $\mathbf{I}_p \in \Re^{p\times p}$ is an identity matrix. Moreover, if the probability density function (pdf) of $\mathbf{X}$ exists, it is of the form

$$f_{EC_p}\left(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, h^{(p)}\right) = |\boldsymbol{\Sigma}|^{-\frac{1}{2}} h^{(p)}\left((\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right), \quad \mathbf{x} \in \Re^p, \tag{1}$$

where $h^{(p)}$ is the density generator function; see [17]. In this case, $\boldsymbol{\varphi}$ can be replaced by $h^{(p)}$ in the above notation, and we denote an elliptical distribution by $\mathbf{X} \sim EC_p\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, h^{(p)}\right)$. Moreover, if $\mathbf{X}$, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are partitioned as

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}, \qquad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix},$$

where $\mathbf{X}_1$ and $\boldsymbol{\mu}_1$ are $q \times 1$ $(q \leq p)$ vectors, $\boldsymbol{\Sigma}_{11}$ is a $q \times q$ positive definite matrix, and so on. Then,

$$\mathbf{X}_1 \sim EC_q\left(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}, h^{(q)}\right),$$
$$\mathbf{X}_2 \mid (\mathbf{X}_1 = \mathbf{x}_1) \sim EC_{p-q}\left(\boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_1), \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{21}, h^{(p-q)}_{w(\mathbf{x}_1)}\right),$$

with $w(\mathbf{x}_1) = (\mathbf{x}_1-\boldsymbol{\mu}_1)^{\mathrm{T}}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_1)$, and $h^{(q)}$ and $h_a^{(p-q)}$ can be expressed in terms of $h^{(p)}$ as

$$h^{(q)}(u) = \frac{\pi^{\frac{p-q}{2}}}{\Gamma\left(\frac{p-q}{2}\right)} \int_0^\infty x^{\frac{p-q}{2}-1} h^{(p)}(u+x)\,\mathrm{d}x, \quad u \geq 0,$$

and

$$h_a^{(p-q)}(u) = \frac{h^{(p)}(u+a)}{h^{(q)}(a)}, \quad u \geq 0.$$

Two important special cases of elliptical distributions are the multivariate normal and t distributions. Specifically, if the generator function in (1) is $h^{(p)}(u) = \mathrm{e}^{-u/2}/(2\pi)^{p/2}$, $u \geq 0$, we get the usual multivariate normal distribution, denoted by $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with pdf

$$\phi_p(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right), \quad \mathbf{x} \in \Re^p,$$

and if for a $\nu > 0$ (degrees of freedom), if the generator function be is

$$h^{(p)}(u) = \frac{\Gamma\left(\frac{\nu+p}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\nu\pi)^{p/2}} \left(1+\frac{u}{\nu}\right)^{-(u+p)/2}, \quad u \geq 0,$$

we get the usual multivariate t distribution, denoted by $\boldsymbol{X} \sim t_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$, with pdf

$$g_p(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu) = \frac{\Gamma\left(\frac{\nu+p}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\nu\pi)^{p/2}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \left(1+\frac{(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{\nu}\right)^{-\frac{\nu+p}{2}}, \ \mathbf{x} \in \Re^p.$$

Since the elliptical distributions are symmetric, multivariate skew-elliptical distributions have been proposed, studied and generalized by many authors. Azzalini [6] discussed formally and popularized the univariate standard skew-normal distribution. A random variable Z_λ is said to have a standard skew-normal distribution with parameter $\lambda \in \Re$, denoted by $Z_\lambda \sim SN(\lambda)$, if its pdf is

$$\phi_{SN}(z; \lambda) = 2\phi(z)\Phi(\lambda z), \quad z \in \Re,$$

where $\phi(z)$ and $\Phi(z)$ denote the standard normal pdf and cdf, respectively. This distribution and its variations have been discussed by several authors including Azzalini [7], Henze [21], Branco and Dey [14], Loperfido [26], Arnold and Beaver [5], Balakrishnan [12], Azzalini and Chiogna [9]. A recent survey of developments on skew-normal distribution and its multivariate form is due to [8].

For the multivariate version, Azzalini and Dalla Valle [10] presented the multivariate skew-normal distribution, while [25] gave a Bayesian interpretation of the multivariate skew-normal distribution.