Statistical Methodology 9 (2012) 211–227

Contents lists available at SciVerse ScienceDirect

Statistical Methodology

journal homepage: www.elsevier.com/locate/stamet

Some theory for anisotropic processes on the sphere

M. Hitczenko *, M.L. Stein

University of Chicago, Department of Statistics, 5734 S. University Ave, Chicago, IL 60637, United States

ARTICLE INFO

Keywords:
Anisotropic covariance functions
Spherical harmonics
Differential operators
Gaussian process
Equivalence and orthogonality

ABSTRACT

We investigate properties of an existing class of models for Gaussian processes on the sphere that are invariant to shifts in longitude. The class is obtained by applying first-order differential operators to an isotropic process and potentially adding an independent isotropic term. For a particular choice of the operators, we derive explicit forms for the spherical harmonic representation of these processes' covariance functions. Because the spherical harmonic representation is a spectral one, these forms allow us to draw conclusions about the local properties of the processes. For one, the coefficients in the spherical harmonic representation relate to the equivalence and orthogonality of the measures induced by the models. It turns out that under certain conditions the models will lack consistent parameter estimability even when the process is observed everywhere on the sphere. We also consider the ability of the models to capture isotropic tendencies on the local level, a phenomenon observed in some data.

1. Introduction: models on the sphere

The analysis of processes on a sphere provides myriad statistical challenges. Advances in technology over the past decade have led to the collection of many geophysical datasets, often gathered via satellite, that are large in size as well as scope. Sea surface temperature (SST) from the TRMM Microwave Imager (TMI) [14], aerosol optical depth (AOD) from the Multi-angle Imaging SpectroRadiometer (MISR) [12], and total column ozone (TCO) from the Total Ozone Mapping Spectrometer (TOMS) [10] are all examples of data that have quite comprehensive global coverage.

Methods of dealing with such data vary and their sensibility depends not only on the nature of the observations, but also on the ultimate goals. For instance, when it comes to spatial interpolation, there is evidence [11] that it is predominantly the observations close to the point of interest that matter, making the division of a dense dataset into smaller subsets appropriate. However, there exist

* Corresponding author.
E-mail addresses: marcin@galton.uchicago.edu, hitczenk@gmail.com (M. Hitczenko), stein@galton.uchicago.edu (M.L. Stein).

1572-3127/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.stamet.2011.01.010

datasets that do not have a high resolution of observations and even those that do often have areas that are relatively sparse. Global models make it possible to deal with such cases. In addition, models which look at the data as a whole easily allow for the study of trends on the mid-range and global scales. The parameters of such models often have some physical intuition behind them and their estimation may give some insight into the geophysical process that led to the observed data. Of course, with the observations taken on such an obviously curved surface, it is impossible to ignore the spherical domain of the data when using such a global approach. Therefore, there is a need to develop models that, besides being suitable for the sphere, manage to strike a balance between complexity and computational efficiency.

Geophysical data consistently show a variety of structures in different parts of the world. For example, many processes, including SST [4], AOD [15], and TCO [3], show great change in both mean and variance with shifts in latitude. In addition, any model should strive to capture various asymmetries on both the large and small scales, such as stronger correlations between observations along certain directions, potentially caused by the direction of winds. To accomplish these goals a great deal of flexibility needs to be afforded.

In this paper, we restrict ourselves to real-valued Gaussian processes $Z(L, l)$ indexed by latitude $L \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and longitude $l \in [-\pi, \pi]$. As a Gaussian process, $Z(L, l)$ will be completely defined in a probabilistic sense by the mean function $m(L, l) = \mathrm{E}Z(L, l)$ and the covariance function $K(L_1, L_2, l_1, l_2) = \mathrm{Cov}(Z(L_1, l_1), Z(L_2, l_2))$.

An assumption of isotropy (or homogeneity) greatly simplifies the modeling of a process on the sphere. A Gaussian isotropic process is defined by a constant mean function and a covariance function that depends only on the great circle distance (or, equivalently, chordal distance) between the two locations. Therefore, on a sphere with radius R, an isotropic process has $m(L, l) = \phi$ for some $\phi \in \mathbb{R}$ and $K(L_1, L_2, l_1, l_2) = f(\mathrm{gcd}(L_1, L_2, l_1 - l_2))$ for some function $f(\cdot)$ with

$$\mathrm{gcd}(L_1, L_2, l_1, l_2) = 2R \arcsin\left(\left\{\sin^2\left(\frac{L_1 - L_2}{2}\right) + \cos L_1 \cos L_2 \sin^2\left(\frac{l_1 - l_2}{2}\right)\right\}^{\frac{1}{2}}\right). \quad (1.1)$$

Unfortunately, the analysis of data from various atmospheric processes has shown that an assumption of isotropy is generally inappropriate [16].

As is apparent from the formula for great circle distance (1.1), the isotropic process is just a particular case of a more general class of models that are stationary in longitude. Such processes, dubbed axially symmetric by Jones [8], are defined by a mean function that depends only on latitude and a covariance function that is a function of the longitudes only through their difference. Thus, $m(L, l) = \phi(L)$ for some function $\phi(\cdot)$ and $K(L_1, L_2, l_1, l_2) = g(L_1, L_2, l_1 - l_2)$ for some function $g(\cdot)$. In this case, we express the covariance function as depending on three arguments: $K(L_1, L_2, l)$.

The axially symmetric class provides a great deal more flexibility than isotropy by allowing for varying behavior at different latitudes. It also permits certain types of asymmetries, which the isotropic model does not, for example by differentiating between $K(L_1, L_2, l)$ and $K(L_1, L_2, -l)$. In practice, the statistical characteristics describing many natural processes show more diversity moving along latitudes than longitudes, making this class of models attractive.

It should be noted that any assumption of stationarity is likely to be false in reality, particularly when dealing with global data where the nonuniform distribution of factors such as land or urban areas can lead to significantly different properties around the globe. Often, one can remove a great deal of the non-stationarity via the mean function or by a local rescaling, perhaps by using a model of the form $Z(L, l) = \beta_0(L, l) + \beta_1(L, l)X(L, l)$ where $\beta_0(L, l), \beta_1(L, l)$ are deterministic functions and $X(L, l)$ is an axially symmetric process. Overall, the assumption of axial symmetry serves as a simplification of a complicated reality which one hopes can give an informative approximation to the truth.

Of course, the issue of picking valid axially symmetric covariance functions that are positive definite on the domain of interest remains. This paper will look at and compare two methods of generating axially symmetric covariance functions: one proposed by Jones [8] and another by Jun and Stein [9,13] and Bolin and Lindgren [2]. The former relies on formulating the process as an expansion of spherical harmonic functions while the latter applies differential operators to an isotropic process. The focus here is primarily on the properties of the differential operator model.