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On a Brownian motion with a hard membrane

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ABSTRACT

Local perturbations of a Brownian motion are considered. As a limit we obtain a non-Markov process that behaves as a reflecting Brownian motion on the positive half line until its local time at zero reaches some exponential level, then changes a sign and behaves as a reflecting Brownian motion on the negative half line until some stopping time, etc.

1. Introduction

Consider a sequence of SDEs

$$dX_\varepsilon(t) = a_\varepsilon(X_\varepsilon(t))dt + dw(t), \quad t \geq 0,\ X_\varepsilon(0) = x, \tag{1}$$

where w is a Wiener process.

We assume that a_ε is an integrable function; this ensures existence and uniqueness of a strong solution to this SDE (Engelbert and Schmidt, 1991, Theorem 4.53).

We also will suppose that the support of a_ε is contained in $[-\varepsilon, \varepsilon]$; X_ε will be interpreted as a local perturbation of a Brownian motion.

Condition $\text{supp}(a_\varepsilon) \subset [-\varepsilon, \varepsilon]$ ensures the weak relative compactness in the space of continuous functions of $\{X_{\varepsilon_n}\}$ for any sequence $\varepsilon_n \to 0$ as $n \to \infty$. Indeed, if $\text{supp}(a_\varepsilon) \subset [-\varepsilon, \varepsilon]$, then it can be seen that $\omega_{X_\varepsilon}(\delta) \leq 2\omega_w(\delta) + 2\varepsilon$ for any $\varepsilon > 0, \delta > 0$, where $\omega_g(\delta) = \sup_{s,t\in[0,T];|s-t|\leq\delta} |g(t) - g(s)|$ is the modulus of continuity (see Lemma 3). The aim of this paper is to discuss possible limits of $\{X_\varepsilon\}$ as $\varepsilon \to 0+$.

If $a_\varepsilon(x) = \varepsilon^{-1}a(\varepsilon^{-1}x)$, then a_ε converges in the generalized sense to $\alpha\delta$, where $\alpha = \int_{\mathbb{R}} a(x)dx$ and δ is the Dirac delta function at 0. In this case (Portenko, 1976, Theorem 8), (Lejay, 2006, Proposition 8) we have convergence in distribution in the space of continuous functions

$$X_\varepsilon \Rightarrow w_\gamma,$$

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where $\gamma = \tanh\alpha$, w_γ is a skew Brownian motion, i.e., a continuous Markov process with transition density

$$p_t(x,y) = \varphi_t(x-y) + \gamma\,\mathrm{sgn}(y)\varphi_t(|x|+|y|), \quad x,y\in\mathbb{R},$$

$\varphi_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}$ is the density of the normal distribution $N(0,t)$.

If $\gamma = 1$ (or $\gamma = -1$), then w_γ is a Brownian motion with reflection into the positive (or negative) half-line.

Assume that $\mathrm{sgn}(x)a_\varepsilon(x) \ge 0$. Then the drift term pushes up when X_ε is on the positive half line and pushes down when X_ε is on the negative half line. If the limit of sequence $\mathbb{1}_{x\ge 0}a_\varepsilon(x)$ as $\varepsilon \to 0+$ is "greater than" delta function, then any limit of X_ε cannot cross through zero and consequently the limit will be a reflecting Brownian motion.

Note that the skew Brownian with $|\gamma| < 1$ has both positive and negative excursions in any neighborhood of hitting 0 with probability 1. Reflected Brownian motion ($\gamma = 1$) does not cross zero if it starts from $x \ge 0$; otherwise if $x < 0$, then it crosses 0 immediately after the hitting. We find a situation when a limit of $\{X_\varepsilon\}$ is an intermediate regime between a reflecting case and a skew Brownian motion. The limit process will be a reflecting Brownian motion in some half line until its local time reaches an exponential random variable. Then it behaves as a Brownian motion with reflection into another half line until its local time reaches another independent exponential random variable, etc. We call such process a Brownian motion with a hard membrane. The corresponding definitions are given in Section 2. We prove the general convergence result in Section 3. As an example we discuss in Section 4 the case $a_\varepsilon(x) = L_\varepsilon\varepsilon^{-1}a(\varepsilon^{-1}x)$, $\mathrm{supp}(a) \subset [-1,1]$, where $L_\varepsilon \to \infty$ as $\varepsilon \to 0+$.

The Brownian motion with a hard membrane can be also obtained as a scaling limit of the Lorentz process in a strip with a reflecting wall at the origin that has small shrinking holes (Nandori and Szasz, 2012).

2. Definitions. Reflecting Brownian motion. Brownian motion with a hard membrane

Recall the definition and properties of the Skorokhod reflection problem, see for ex. Pilipenko (2014).

Definition 1. Let $f \in C([0,T])$, $f(0) \ge 0$. A pair of continuous functions g and l is said to be a solution of the Skorokhod problem for f if

S1. $g(t) \ge 0,\ t\in[0,T]$;
S2. $g(t) = f(t) + l(t),\ t\in[0,T]$;
S3. $l(0) = 0,\ l$ is non-decreasing;
S4. $\int_0^T \mathbb{1}_{g(s)>0}dl(s) = 0$.

It is well known that there exists a unique solution to the Skorokhod problem and the solution is given by the formula

$$l(t) = -\min_{s\in[0,t]}(f(s)\wedge 0) = \max_{s\in[0,t]}(-f(s)\vee 0), \tag{2}$$

$$g(t) = f(t) + l(t) = f(t) - \min_{s\in[0,t]}(f(s)\wedge 0). \tag{3}$$

We will say that g is a process reflecting into a positive half line and denote it by $f^{\mathit{refl},+}$.

Remark 1. If $f(0) < 0$, then we set by definition $f^{\mathit{refl},+}(t) := f(t)$ until the instant ζ_0 of hitting 0, and $f^{\mathit{refl},+}(t) := f(t) - \min_{s\in[\zeta_0,t]} f(s)$ for $t \ge \zeta_0$.

The reflection problem with reflection into the negative half line is constructed similarly, $g(t) := f(t) - l(t)$, where l is also non-decreasing. In this case denote g by $f^{\mathit{refl},-}$.

Let $w(t), t \ge 0$ be a Brownian motion started at x, $w^{\mathit{refl},+}$ and $w^{\mathit{refl},-}$ be reflecting Brownian motions with reflection at 0 into the positive and negative half lines, correspondingly. It is known that the process $l(t)$ is the two-sided local time of $w^{\mathit{refl},\pm}$ at 0 defined by

$$\lim_{\varepsilon\to 0+}(2\varepsilon)^{-1}\int_0^t \mathbb{1}_{[-\varepsilon,\varepsilon]}(w^{\mathit{refl},\pm}(s))ds.$$

Consider two sequences of exponential random variables $\{\xi_k^+\}$ and $\{\xi_k^-\}$ with parameters $\alpha^\pm$, respectively. Suppose that all random variables $\{\xi_k^\pm\}$ and the Brownian motion $w(t), t \ge 0$ are mutually independent.

Assume that $w(0) = x > 0$. The Brownian motion with a hard membrane and parameters of permeability $\alpha^\pm$ is constructed in the following way.

$$w^{hard}(t) := w^{\mathit{refl},+}(t) \quad \text{if } l(t) \le \xi_1^+.$$

At the moment $\zeta_1^+ := \inf\{t \ge 0 \mid l(t) \ge \xi_1^+\}$ the process w^{hard} changes orientation. It reflects into negative half line until the moment when its local time after ζ_1^+ reaches the level ξ_1^-:

$$w^{hard}(t) := w(t) - w(\zeta_1^+) - (l(t) - l(\zeta_1^+)) = w(t) - w(\zeta_1^+) - \max_{s\in[\zeta_1^+,t]}(w(s) - w(\zeta_1^+))$$

for $t \le \inf\{s \ge \zeta_1^+ : (l(s) - l(\zeta_1^+)) \ge \xi_1^-\} = \inf\{t \ge \zeta_1^+ : \max_{s\in[\zeta_1^+,t]}(w(s) - w(\zeta_1^+)) \ge \xi_1^-\}$.