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Randomly stopped sums of not identically distributed heavy tailed random variables

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ABSTRACT

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent but not necessarily identically distributed non-negative random variables a number of which has distribution functions with dominatingly varying tails. Let η be a counting random variable independent of $\{\xi_1, \xi_2, \ldots\}$. We consider conditions for random variables $\{\xi_1, \xi_2, \ldots\}$ and η under which distribution of the random sum $\xi_1 + \xi_2 + \cdots + \xi_\eta$ preserves dominatingly varying tail.

1. Introduction

Let $\{\xi_1, \xi_2, \ldots\}$ be independent random variables (r.v.s) having distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. We call r.v. η a *counting* r.v. if it is nonnegative, integer valued and non-degenerate at zero. We denote $S_0 = 0$, $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geqslant 1$ and $S_\eta = \xi_1 + \xi_2 + \cdots + \xi_\eta$. In this paper, we analyze in what cases d.f. of the randomly stopped sum

$$\begin{aligned} F_{S_\eta}(x) &= \mathbb{P}(S_\eta \leqslant x) = \sum_{n=0}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n \leqslant x) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(\eta = n)\, F_{\xi_1} * F_{\xi_2} * \ldots * F_{\xi_n}(x) \end{aligned} \tag{1.1}$$

belongs to the class $\mathcal{D}$ by supposing that r.v.s $\{\xi_1, \xi_2, \ldots\}$ and η satisfy suitable conditions. The assertions below generalize some results of the paper Leipus and Šiaulys (2012) in which the case of identically distributed $\{\xi_1, \xi_2, \ldots\}$ were considered.

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In our considerations r.v.s $\{\xi_1, \xi_2, \dots\}$ are not necessarily identically distributed. Clearly, the asymptotic behavior of d.f. F_{S_η} is closely related to the behavior of d.f.s $\{F_{\xi_1}, F_{\xi_2}, \dots\}$.

Before discussing the properties of F_{S_η}, we recall the definitions of some classes of heavy-tailed d.f.s. We remark only that $\overline{F}(x) = 1 - F(x)$ for all real x and an arbitrary d.f. F.

- *A d.f. F is said to be heavy-tailed* $(F \in \mathcal{H})$ *if for every fixed* $\delta > 0$, *we have*

$$\lim_{x\to\infty} \overline{F}(x)\mathrm{e}^{\delta x} = \infty.$$

- *A d.f. F is long-tailed* $(F \in \mathcal{L})$ *if for every positive* y *(equivalently, for some* $y > 0$*)*

$$\overline{F}(x+y) \underset{x\to\infty}{\sim} \overline{F}(x).$$

- *A d.f. F has dominatingly varying tail* $(F \in \mathcal{D})$ *if for every fixed* $y \in (0, 1)$ *(equivalently, for some* $y \in (0, 1)$*)*

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

- *A d.f. F supported on the interval* $[0, \infty)$ *is subexponential (belongs to the class* $\mathcal{S}$*) if*

$$\lim_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2.$$

If d.f. F is supported in $\mathbb{R}$, then F belongs to one of classes $\mathcal{H}$, $\mathcal{L}$, $\mathcal{D}$ or $\mathcal{S}$ if d.f. $F^+(x) = F(x)\mathbb{1}_{[0,\infty)}(x)$ belongs to the corresponding class. It is known (see, e.g., Embrechts and Omey, 1984, Klüppelberg, 1988, or Chapters 1.4 and A3 in Embrechts et al., 1997) that

$$\mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}, \quad \mathcal{D} \subset \mathcal{H}.$$

There exist many results about sufficient or necessary and sufficient conditions in order that the d.f. of random sum (1.1) belongs to a heavy-tailed distribution classes. Usually, in such results, it is assumed that r.v.s $\{\xi_1, \xi_2, \dots\}$ are not only independent but also identically distributed. Below we present a few known results. The first classical statement is about the closure of the randomly stopped sum of the subexponentially distributed r.v.s.

Theorem 1.1. *Suppose that* $\{\xi_1, \xi_2, \dots\}$ *are independent copies of a nonnegative r.v.* ξ *with d.f.* $F_\xi \in \mathcal{S}$. *Let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \dots\}$. *If* $\mathbb{E}(1+\delta)^\eta < \infty$ *for some* $\delta > 0$, *then* $F_{S_\eta} \in \mathcal{S}$ *as well.*

The proof of Theorem 1.1 can be found in several papers, e.g., in Embrechts and Goldie (1982) (see Theorem 4.2), Cline (1987) (see Theorem 2.13), Embrechts et al. (1997) (see Theorem 1.3.9 and A3.20), Foss et al. (2011) (see Corollary 3.13 and Theorem 3.37). In a more general case, where F_ξ belongs to the so-called convolution equivalent class $\mathcal{S}(\alpha)$, $\alpha \geqslant 0$, a similar result was obtained by Watanabe (2008).

Analogous property for distributions with dominatingly varying tails can be found in Leipus and Šiaulys (2012) (see Theorems 4, 5 and Corollary 1). Below we formulate Theorem 4 of this paper. We recall only that d.f. F belongs to the class $\mathcal{D}$ if and only if the upper Matuszewska index $J_F^+ < \infty$, where, by definition,

$$J_F^+ = -\lim_{y\to\infty} \frac{1}{\log y} \log\left(\liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}\right).$$

Theorem 1.2. *Let* $\{\xi_1, \xi_2, \dots\}$ *be i.i.d. nonnegative r.v.s with common d.f.* $F_\xi \in \mathcal{D}$. *Let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \dots\}$ *such that* $\mathbb{E}\eta^{p+1} < \infty$ *for some* $p > J_{F_\xi}^+$. *Then d.f.* F_{S_η} *of the randomly stopped sum* S_η *belongs to the class* $\mathcal{D}$ *as well.*

The closeness of the class $\mathcal{L}$ under the random convolution was considered by Cline (1987, 1990), Albin (2008) and Leipus and Šiaulys (2012). Below we present the statement of Theorem 6 from Leipus and Šiaulys (2012).

Theorem 1.3. *Suppose that* $\{\xi_1, \xi_2, \dots\}$ *are i.i.d. nonnegative r.v.s with d.f.* $F_\xi \in \mathcal{L}$. *Let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \dots\}$ *with d.f.* F_η. *If* $\overline{F_\eta}(\delta x) = o\big(\sqrt{x}\,\overline{F_\xi}(x)\big)$ *for any* $\delta \in (0, 1)$ *then* $F_{S_\eta} \in \mathcal{L}$.

In this paper, we generalize Theorem 1.2 for the inhomogeneous case. We consider the situation when r.v.s $\{\xi_1, \xi_2, \dots\}$ are independent, but not necessary identically distributed. We suppose that some r.v.s from $\{\xi_1, \xi_2, \dots\}$ have distributions belonging to the class $\mathcal{D}$ and we find the minimal conditions for r.v.s $\{\xi_1, \xi_2, \dots\}$ and η in order that distribution of the