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Nonparametric probability weighted empirical characteristic function and applications

Simos G. Meintanis *, Nikolai G. Ushakov

Department of Economics, National and Kapodistrian University of Athens, Athens, Greece
Unit for Business Mathematics and Informatics, North–West University, Potchefstroom, South Africa[1]
Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim, Norway

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ABSTRACT

We study certain properties of probability weighted quantities and suggest a nonparametric estimator of the probability weighted characteristic function introduced by Meintanis et al. (2014). Some properties of this estimator are studied and corresponding inference procedures for symmetry and the two-sample problem are proposed. Monte Carlo results on the finite-sample behavior of the procedures are also included.

1. Introduction

The empirical characteristic function (ECF) has been for many years a basic tool of statistical inference. Recent works on the ECF include, Jiménez–Gamero et al. (2015), Pardo-Fernández et al. (2015), Henze et al. (2014), Potgieter and Genton (2013), Leucht (2012), Quessy and Éthier (2012), and Tenreiro (2011), just to name a few. For review articles of the work on the ECF the reader is referred to Csörgő (1984), Epps (2005), Hušková and Meintanis (2008a,b), and to the book by Ushakov (1999). Recently Meintanis et al. (2014) introduced the notion of the probability weighted ECF (PWECF), which offers some sort of adaptiveness of ECF-based procedures. The idea of the PWECF draws from earlier works in which probability weighted quantities are employed in the context of estimation of parameters; see for instance, Diebolt et al. (2004), Caeiro and Gomes (2011), and Caeiro et al. (2014). The drawback of this PWECF however is that it was defined under a very strict parametric model. Therefore it is not an appropriate tool when dealing with nonparametric or semiparametric inference where the corresponding model domain is much wider. The purpose of this paper is to suggest a nonparametric version of the PWECF and use this quantity for statistical inference. In this connection, our primary examples will be the construction of tests for symmetry and for the two-sample problem. The remainder of this work is outlined as follows. In Section 2 the probability weighted quantities are introduced and some of their properties are studied. In Section 3 we suggest goodness-of-fit test procedures which are based on appropriate functionals of the nonparametric PWECF. In Section 4 some computational

* Corresponding author at: Department of Economics, National and Kapodistrian University of Athens, Athens, Greece.
E-mail address: simosmei@econ.uoa.gr (S.G. Meintanis).

[1] On sabbatical leave from the University of Athens.

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details are provided while finite-sample properties of the procedures are illustrated by means of a simulation study in Section 5. All proofs are collected in an Appendix.

2. Probability weighted distributions and characteristic functions

In this section we introduce some general definitions regarding probability weighted quantities and obtain properties for these quantities as well as for their corresponding estimators. In what follows we write X for an arbitrary random variable and denote by $F(x)$ and $\varphi(t)$ the corresponding distribution function (DF) and characteristic function (CF). Also $X_1, \dots, X_n$, denote independent copies of the random variable X.

2.1. Probability weighted distribution function and its empirical counterpart

A probability weighted moment (PWM) of X (or of $F(x)$) is defined as

$$\mu_{m,r,s} = \mathrm{E}[X^m(F(X)^r)(1-F(X))^s]$$

where m is a nonnegative integer, and r and s, are nonnegative real numbers. PWMs were first used in order to estimate parameters of certain distributions; see for instance Greenwood et al. (1979). Although these authors use a general definition with arbitrary values of (r, s), in what follows we will consider only the case when $r = s$.

As it turns out, any PWM of $F(x)$ is an ordinary moment of another DF corresponding to $F(x)$, which is generalized, i.e., it is a DF of a non-normalized distribution. This last distribution is defined below.

Definition 2.1. The function

$$F^{[s]}(x) = \int_{-\infty}^{x} [F(u)(1-F(u))]^s dF(u), \quad s \geq 0, \tag{2.1}$$

is called a probability weighted DF (PWDF) of $F(x)$ (or of X).

If the change of variable $v = F(u)$ can be done in (2.1), in particular, when the DF is continuous and strictly monotone, then we get another representation of $F^{[s]}(x)$:

$$F^{[s]}(x) = \int_{0}^{F(x)} [v(1-v)]^s dv. \tag{2.2}$$

This representation allows one to prove some results easier. In what follows we often formulate results in the general case while give proofs in the case when the integrals in (2.1) and (2.1) are equal. Proofs in the general case are obtained using standard approximation procedures: each DF is approximated by a convergent sequence of continuous and strictly increasing DFs.

Let $s \geq 0$, and write

$$\tau_s = \int_{0}^{1} [v(1-v)]^s dv. \tag{2.3}$$

The following theorem shows that PWMs of $F(x)$ are ordinary moments of the corresponding PWDF. The proof of this theorem as well as the proofs of Theorems 2.2 and 2.4 of this subsection are quite simple and therefore omitted.

Theorem 2.1. *For any fixed $s \geq 0$, $F^{[s]}(x)$ is a generalized (non-normalized) distribution function, i.e., it is non-decreasing, right continuous, and satisfies*

$$\lim_{x\to-\infty} F^{[s]}(x) = 0, \qquad \lim_{x\to\infty} F^{[s]}(x) = \tau_s,$$

where the constant τ_s is given by (2.3). Moreover the ordinary moments of $F^{[s]}(x)$ coincide with the PWM's of $F(x)$, that is

$$\mu_{m,s} = \mathrm{E}[X^m(F(X)(1-F(X)))^s] = \int_{-\infty}^{\infty} x^m dF^{[s]}(x).$$

Remark 2.1. It is easy to see that if $F(x)$ is absolutely continuous with the density $f(x)$, then, for any $s \geq 0$, $F^{[s]}(x)$ is absolutely continuous with density

$$f^{[s]}(x) = [F(x)(1-F(x))]^s f(x).$$

In the following theorem it is shown that the PWDF $F^{[s]}(x)$ uniquely determines the DF $F(x)$.