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# Stochastic functional differential equations of Sobolev-type with infinite delay[☆]

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ABSTRACT

Stochastic differential equations have been widely used to model a number of phenomena in diverse fields of science and engineering. In this paper, we focus on the local existence of mild solution for a class of stochastic functional differential equations of Sobolev-type with infinite delay. Furthermore, the results are extended to study the local existence results for neutral stochastic differential equations of Sobolev-type. Using the semigroup theory and fixed point argument, we establish a set of sufficient conditions for obtaining the required result. Furthermore, existence results for integro-differential equations of Sobolev-type is also discussed. Finally, an example is provided to illustrate the obtained theory.


## 1. Introduction

A natural extension of a deterministic differential equation model is a system of stochastic differential equation, where relevant parameters are modeled as suitable stochastic processes. This is due to the fact that most problems in a real life situation are basically modeled by stochastic equations rather than deterministic. Furthermore, it should be mentioned that noise or stochastic perturbation is unavoidable in nature as well as in man-made systems. Thus, stochastic differential equations have attracted great interest due to their extensive applications in describing many sophisticated dynamical systems in physical, biological, medical and social sciences; one can see Diop et al. (2014), El-Borai et al. (2010), Sakthivel et al. (2013), Yan and Yan (2013), Zhu (2014b), Zhu (2014a) and Zhu (2011) and the references therein for details. On the other hand, the Sobolev-type differential equations arise naturally in the mathematical modeling of various physical phenomena such as in the fluid flow through fissured rocks, thermodynamics, shear in second order fluids and so on (see Brill, 1977, Keck and Mckibben, 2003 and Lightbourne and Rankin, 1983 and the references therein). Showalter (Showalter, 1972) established the existence of solutions for a semilinear evolution equation in Banach space which is modeled on boundary value problems for partial differential equations of Sobolev type. Many authors have studied the existence, uniqueness, continuation and other properties of solutions of various special forms of the deterministic Sobolev-type differential equations by using different techniques (see Fečkan et al., 2013, Kucche and Dhakne, 2014, Li et al., 2012 and Mahmudov, 2013 and the references therein).

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However, there are only few papers on Sobolev-type differential equations in stochastic setting. Keck and Mckibben (2003) studied the global existence and uniqueness of mild solutions for a class of first-order abstract stochastic Sobolev-type integro-differential equations in a real separable Hilbert space in which they allow the nonlinearities at a given time $t$ to depend not only on the state of the solution at time but also on the corresponding probability distribution at time $t$. Li et al. (2012) obtain the existence results for fractional evolution equations of Sobolev type by virtue of the theory of propagation family via the techniques of the measure of non compactness and the condensing maps. On the other hand, neutral stochastic differential equations with infinite delay have become important in recent years as mathematical models of phenomena in both physical and social sciences. However, it should be emphasized that to the best of our knowledge the local existence results for stochastic functional differential equations of Sobolev-type with infinite delay in phase space $\mathcal{B}_h$ has not been investigated yet and aim of this paper is to fill this gap.

Throughout this paper, let $(\mathbb{H}, \|\cdot\|)$ and $(\mathbb{K}, \|\cdot\|)$ be two real separable Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{L}(\mathbb{K},\mathbb{H})$ be the space of bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. In the sequel, without confusion, we also employ the inner product and the norm denoted, respectively, by $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ for $\mathcal{L}(\mathbb{K},\mathbb{H})$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. Let $W = (W_t)_{t\geq 0}$ represent a $Q$-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the covariance operator $Q$ such that $TrQ < \infty$. Furthermore, we assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers $\lambda_k$ such that $Qe_k = \lambda_k e_k,\ k = 1, 2, \ldots,$ and a sequence of independent Wiener processes $\{\beta_k\}_{k\geq 1}$ such that

$$\langle w(t), e\rangle_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle e_k, e\rangle_{\mathbb{K}}\beta_k(t), \quad e \in \mathbb{K},\ t \geq 0.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K},\mathbb{H})$ be the space of all Hilbert–Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with the inner product $\langle\varphi,\psi\rangle_{\mathcal{L}_2^0} = Tr[\varphi Q\psi*]$ (see Cui and Yan, 2011 and the references therein).

In this paper, we consider the stochastic functional differential equations of Sobolev-type with infinite delay in the following form

$$\begin{cases} d[Bx(t)] = [Ax(t) + f(t, x_t)]dt + \sigma(t, x_t)dW(t), & t \in I := [0, T], \\ x(t) = \phi(t) \in \mathcal{B}_h, & t \in (-\infty, 0], \end{cases} \tag{1.1}$$

where the state $x(\cdot)$ takes values in the separable real Hilbert space $\mathbb{H}$; $A$ and $B$ are linear operators on $\mathbb{H}$. The histories $x_t : (-\infty, 0] \to \mathcal{B}_h, x_t(\theta) = x(t+\theta)$ for $t \geq 0$ belongs to the phase space $\mathcal{B}_h$, which will be defined later. The initial data $\phi = \{\phi(t),\ t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}_h$-valued stochastic process independent of $W$ with finite second moments. Furthermore, $f : I \times \mathcal{B}_h \to \mathbb{H}$ and $\sigma : I \times \mathcal{B}_h \to \mathcal{L}_2^0(\mathbb{K},\mathbb{H})$ are appropriate mappings specified later.

The main purpose of this work is to extend the results concerning the local existence of mild solutions of various classes of deterministic Sobolev-type evolution equations discussed in Kucche and Dhakne (2014) and Li et al. (2012) to a more general class of stochastic functional evolution equations of the abstract form (1.1). Equations of the form (1.1) and their special forms may arise in various physical phenomena such as in flow of fluid through fissured rocks, the propagation of long waves of small amplitudes, thermodynamics and shear in second order fluids. In the present paper, we prove the existence of mild solutions for (1.1) using the fixed point technique and semigroup theory.

## 2. Main results

In this section, we study the existence results for the stochastic functional differential equations of Sobolev-type with infinite delay. In particular, we formulate and prove a new set of conditions for the existence of mild solutions for stochastic functional differential equations of Sobolev-type (1.1) by means of fixed point theorem of nonlinear alternative Leray–Schauder type. Then, the result is extended to study the neutral stochastic differential equations of Sobolev-type with the help of Sadovskii's fixed point theorem and semigroup theory.

Now, we present some preliminaries which are needed to establish main results. For details on this section, the reader may refer to Cui and Yan (2011), Kucche and Dhakne (2014) and Ren and Sakthivel (2012) and the references therein. For the problem (1.1), the operators $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ and $B : D(B) \subset \mathbb{H} \to \mathbb{H}$ satisfy the following conditions (Lightbourne and Rankin, 1983):

(E1) $A$ and $B$ are closed linear operators,
(E2) $D(B) \subset D(A)$ and $B$ is bijective,
(E3) $B^{-1} : \mathbb{H} \to D(B)$ is continuous.

Furthermore, from (E1) and (E2) $B^{-1}$ is closed then with (E3) by using the closed graph theorem, we obtain the boundedness of the linear operator $AB^{-1} : \mathbb{H} \to \mathbb{H}$. Furthermore, $AB^{-1}$ generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in $\mathbb{H}$. Let us denote $\max_{t\in I}\|T(t)\|^2 = M,\ \|B^{-1}\|^2 = M_B$ and $\|B\|^2 = \widetilde{M}_B$.

Now, we present the abstract phase space $\mathcal{B}_h$ (Cui and Yan, 2011). Assume that $h : (-\infty, 0] \to (0,\infty)$ with $l = \int_{-\infty}^{0} h(t)dt < \infty$ a continuous function and $\phi$ is $\mathcal{F}_0$-measurable functions mapping from $(-\infty, 0]$ into $\mathbb{H}$. Define