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Weak laws of large numbers for weighted independent random variables with infinite mean

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ABSTRACT

We study weak laws of large numbers for weighted independent random variables with infinite mean. In particular, this paper explores the case that the decay order of the tail probability is -1. Moreover, we extend a result concerning the Pareto–Zipf distributions given by Adler (2012).

1. Introduction

Notation: Symbols $a_n \sim b_n$, $a_n \asymp b_n$ and $a_n = o(b_n)$ stand for $\lim a_n/b_n = 1$, $0 < \liminf a_n/b_n \le \limsup a_n/b_n < \infty$ and $\lim a_n/b_n = 0$, respectively. We redefine the natural logarithm $\log x$ as the meaning of $\log(\max\{x, e\})$ for $x > 0$. The indicator random variable is defined by $\mathbf{1}(A) := \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A \end{cases}$ for each event A.

We consider independent and identically distributed (i.i.d.) random variables to carry out limiting procedures. When they have a common finite mean, the ordinary laws of large numbers are formulated by the sample average. If the mean is infinite, some devices are needed. In special cases by adjusting scaling parameters we can obtain a similar result of the ordinary law of large numbers. We also call it a law of large numbers. The typical example is known as the *St. Petersburg game*, which is written in Sec. X.4 Eq. (4.1) of Feller (1968). A similar example is less known as the *Feller game*, which is based on the first return time to the origin of simple random walks (see Thm. 3 (i) of Matsumoto and Nakata, 2013). These games are formulated by a nonnegative random variable X with the tail probability $\mathbb{P}(X > x) \asymp x^{-\alpha}$ for each fixed $0 < \alpha \le 1$. In addition, considering truncated random variables Nakata (2015) studied strong laws of large numbers and central limit theorems in this situation.

On the other hand, Adler (1990a) investigated laws of large numbers for *weighted* sums of i.i.d. random variables. Moreover, Adler (2007, 2008) obtained related results specifying asymmetric Cauchy and two-tailed Pareto distributions, respectively. While the i.i.d. property is assumed in these results, Adler (2012) got rid of the identically distributed condition with respect to independent Pareto–Zipf distributions. He studied weighted laws of large numbers for each model respectively.

Let us consider a random variable satisfying that

$$\mathbb{P}(|X| > x) \asymp x^{-\alpha} \quad \text{for a fixed } 0 < \alpha \le 1. \tag{1}$$

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In this paper, we investigate weak laws of large numbers for weighted independent random variables with Eq. (1) (see Theorem 2.1). Using Theorem 2.1 with $\alpha = 1$, in Theorems 3.1 and 3.2, we extend the result given by Adler (2012). Now, it is known that a weak law holds but a strong law fails for the St. Petersburg game (see Ex. 4 of Adler, 1990b). A similar situation occurs for the setting of independent Pareto–Zipf distributions in Thm. 2 of Adler (2012). We then state that the same facts hold in our setting in Remark 3.1. Moreover, in Theorem 4.1, we give a simple claim of weak laws for Eq. (1) with $\alpha = 1$ under some strong conditions including the i.i.d. property. Using Theorem 4.1, we can avoid checking individual conditions like Thm. 3.1 of Adler (2007) and Thm. 3.1 of Adler (2008). Corollary 4.1 is a handy statement rather than Theorem 4.1. In virtue of Corollary 4.1 or Theorem 4.1 we easily obtain weak laws of large numbers in Section 4.1.

2. Weak laws of large numbers

If a random variable X fulfills Eq. (1), then $\mathbb{E}|X| = \infty$. Moreover, we also obtain the rough asymptotics of the first and the second truncated moments as follows.

Lemma 2.1. *For a random variable X with Eq. (1), we have*

$$\mathbb{E}(|X|\mathbf{1}(|X| \le x)) \asymp \begin{cases} x^{1-\alpha}, & \text{if } 0 < \alpha < 1, \\ \log x, & \text{if } \alpha = 1, \end{cases} \tag{2}$$

and

$$\mathbb{E}(|X|^2\mathbf{1}(|X| \le x)) \asymp x^{2-\alpha} \quad \text{for } 0 < \alpha \le 1. \tag{3}$$

Proof. This is essentially the same as the proof of Prop. 2.1 of Nakata (2015). □

Lemma 2.2. *Let $\{X_j\}$ be independent random variables whose distributions satisfying $\mathbb{P}(|X_j| > x) \asymp x^{-\alpha}$ for $j \ge 1$ and $\limsup_{x\to\infty} \sup_{j\ge 1} x^{\alpha}\mathbb{P}(|X_j| > x) < \infty$. Moreover, we assume that positive sequences $\{a_j\}$ and $\{b_n\}$ satisfy*

$$a_j > 0, \qquad b_n > 0 \quad \text{and} \quad \sum_{j=1}^{n} a_j^{\alpha} = o(b_n^{\alpha}). \tag{4}$$

Then we have

$$\lim_{n\to\infty} \sum_{j=1}^{n} \mathbb{P}(|X_j| > b_n/a_j) = 0 \tag{5}$$

and

$$\lim_{n\to\infty} \frac{1}{b_n^2} \sum_{j=1}^{n} a_j^2 \mathbb{E}X_j^2 \mathbf{1}\{|X_j| \le b_n/a_j\} = 0. \tag{6}$$

Proof. Since Eqs. (1) and (4) demonstrate for some $M > 0$

$$\sum_{j=1}^{n} \mathbb{P}(|X_j| > b_n/a_j) \le M b_n^{-\alpha} \sum_{j=1}^{n} a_j^{\alpha} \to 0 \quad \text{as } n \to \infty.$$

Eq. (5) holds. Using Eqs. (3) and (4), we have for some $M' > 0$

$$\begin{aligned} \frac{1}{b_n^2} \sum_{j=1}^{n} a_j^2 \mathbb{E}X_j^2 \mathbf{1}\{|X_j| \le b_n/a_j\} &\le \frac{M'}{b_n^2} \sum_{j=1}^{n} a_j^2 (b_n/a_j)^{2-\alpha} \\ &\asymp b_n^{-\alpha} \sum_{j=1}^{n} a_j^{\alpha} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence, Eq. (6) follows. □

Theorem 2.1. *Under the assumptions of Lemma 2.2, we have*

$$\lim_{n\to\infty} b_n^{-1} \sum_{j=1}^{n} a_j \left(X_j - \mathbb{E}X_j \mathbf{1}\left\{ |X_j| \le \frac{b_n}{a_j} \right\} \right) = 0 \quad \text{in probability}. \tag{7}$$