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New results for tails of probability distributions according to their asymptotic decay

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ABSTRACT

This paper provides new properties for tails of probability distributions belonging to a class defined according to the asymptotic decay of the tails. This class contains the one of regularly varying tails of distributions. The main results concern the relation between this larger class and the maximum domains of attraction of Fréchet and Gumbel.

1. Introduction

An extension of the class *RV* of regularly varying (RV) functions has been introduced and analyzed in detail in a recent study (Cadena and Kratz, 2015). The characteristic properties of this new larger class allow one, not only to extend main RV properties as described in Cadena and Kratz (2015), but also to deepen the understanding of some Tauberian theorems (Cadena, 2015a) and to build in a simple way an estimator of the tail index on this class, and consequently on the class *RV*, with a good rate of convergence (Cadena, 2015b).

In this paper, we focus on the probabilistic side of this large class, providing new results for tails of distributions belonging to it, according to the asymptotic decay of the tails.

Let us briefly introduce the definition of the sets that will be considered and recall their characteristic properties.

Let $\mathcal{M}$ be the class of measurable functions $U : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\exists \rho \in \mathbb{R}, \quad \forall \varepsilon > 0, \qquad \lim_{x\to\infty} \frac{U(x)}{x^{\rho+\epsilon}} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{U(x)}{x^{\rho-\epsilon}} = \infty \tag{1}$$

where ϵ may be taken arbitrarily small. It can be proved that, for any $U \in \mathcal{M}$, ρ defined in (1) is unique; it is denoted by ρ_U and called the $\mathcal{M}$-index of U.

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Any function U of $\mathcal{M}$, satisfies the following (Cadena and Kratz, 2015, Theorem 1.1, with a minor modification in the notation):

$$U \in \mathcal{M} \quad \text{with} \quad \rho_U = \tau \iff \lim_{x\to\infty} \frac{\log(U(x))}{\log(x)} = \tau \tag{2}$$

where ρ_U is defined in (1).

Combining this characterization (2) with Theorem 1 in Karamata (1933) provides (Cadena and Kratz, 2015, Proposition 2.1) that

$$RV \subsetneq \mathcal{M}$$

and that for any RV function $U \in RV_\alpha$, its tail index α coincides with its $\mathcal{M}$-index ρ_U defined in (1):

$$U \in RV_\alpha \;\Rightarrow\; U \in \mathcal{M} \quad \text{with } \mathcal{M}\text{-index}\, \alpha. \tag{3}$$

Recall, for completeness, that a measurable function $U : \mathbb{R}^+ \to \mathbb{R}^+$ is RV_α (see Karamata, 1930 and e.g. Bingham et al., 1989) if, for some $\alpha \in \mathbb{R}$ called the tail index of U, and for any $t > 0$,

$$\lim_{x\to\infty} \frac{U(tx)}{U(x)} = t^\alpha. \tag{4}$$

We also introduce a natural extension of $\mathcal{M}$ defined in Cadena and Kratz (2015) (with a small modification in the notation) by

$$\mathcal{M}_{-\infty} = \left\{ U : \mathbb{R}^+ \to \mathbb{R}^+ : \forall \rho \in \mathbb{R},\ \lim_{x\to\infty} \frac{U(x)}{x^\rho} = 0 \right\}.$$

As for $\mathcal{M}$, we have a characterization for $\mathcal{M}_{-\infty}$, namely (Cadena and Kratz, 2015, Theorem 1.4)

$$U \in \mathcal{M}_{-\infty} \iff \lim_{x\to\infty} \frac{\log(U(x))}{\log(x)} = -\infty \tag{5}$$

for any positive measurable function U with support $\mathbb{R}^+$.

In view of (2) and (5), we see that $\mathcal{M}$ and $\mathcal{M}_{-\infty}$ allow one to sort the tails of distributions F by their behavior as $x \to \infty$, $\mathcal{M}$ including the tails of distributions with an asymptotic polynomial decay and $\mathcal{M}_{-\infty}$ the ones with an asymptotic exponential behavior.

For the tails of distributions which have neither a polynomial nor an exponential behavior, we introduce another class $\mathcal{O}$, namely (Cadena and Kratz, 2015)

$$\mathcal{O} := \{U : \mathbb{R}^+ \to \mathbb{R}^+ : \mu(U) < \nu(U)\} \tag{6}$$

where $\mu(U)$, $\nu(U)$ correspond to the *lower order* of U and *upper order* of U, respectively, defined by (see for instance Bingham et al., 1989, pp. 73)

$$\mu(U) = \underline{\lim}_{x\to\infty} \frac{\log(U(x))}{\log(x)} \quad \text{and} \quad \nu(U) = \overline{\lim}_{x\to\infty} \frac{\log(U(x))}{\log(x)}.$$

The class $\mathcal{O}$ is non empty, as shown in Cadena and Kratz (2015) where we provided explicit examples. Moreover $\mathcal{M}$, $\mathcal{M}_{-\infty}$, and $\mathcal{O}$ are disjoint sets.

In this study, our main object of interest is tails of distributions F satisfying $x^* = \sup\{x : F(x) < 1\} = \infty$. We denote $\overline{F} = 1 - F$ and will use abusively this notation when referring to tails of distributions.

The paper is organized as follows. Section 2 provides the main properties for tails of distributions belonging to $\mathcal{M}$ and $\mathcal{M}_{-\infty}$. The results given in Section 3 concern the maximum domains of attraction to $\mathcal{M}$ and $\mathcal{M}_{-\infty}$, followed by conclusions in the last section.

All along the paper, we use the following notation: $\min(a, b) = a \wedge b$, $\max(a, b) = a \vee b$, $\lfloor x \rfloor$ for the largest integer not greater than x, $\lceil x \rceil$ for the lowest integer greater than or equal to x, and $\log(x)$ for the natural logarithm of x.

2. Properties for tails of distributions belonging to $\mathcal{M}$ and $\mathcal{M}_{-\infty}$

Let us summarize the main properties (Cadena and Kratz, 2015, Properties 1.1 to 1.4, and Remark 1.1) when considering tails of distribution belonging to the two classes $\mathcal{M}$ or $\mathcal{M}_{-\infty}$. We refer also to Cadena and Kratz (2015) for the proofs.

Let F and G be two distributions.

1. For $(\overline{F}, \overline{G}) \in \mathcal{M} \times \mathcal{M}$ s.t. $\rho_{\overline{F}} > \rho_{\overline{G}}$, or $(\overline{F}, \overline{G}) \in \mathcal{M} \times \mathcal{M}_{-\infty}$, we have $\lim_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} = 0$.
2. Let $\overline{F} \in \mathcal{M}$, bounded on finite intervals. If $\rho_{\overline{F}} < -1$, then $\overline{F}$ is integrable on $\mathbb{R}^+$, whereas, if $\rho_{\overline{F}} > -1$, $\overline{F}$ is not integrable on $\mathbb{R}^+$. Note that in the case $\rho_{\overline{F}} = -1$, we can find examples of functions $\overline{F}$ which are integrable or not.