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Intermittency for the wave equation with Lévy white noise

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ABSTRACT

In this article, we consider the stochastic wave equation on $\mathbb{R}_+ \times \mathbb{R}$ driven by the Lévy white noise introduced in Balan (2015). Using Rosenthal's inequality, we develop a maximal inequality for the moments of order $p \geq 2$ of the integral with respect to this noise. Based on this inequality, we show that this equation has a unique solution, which is weakly intermittent in the sense of Foondun and Khoshnevisan (2009) and Khoshnevisan (2014).

1. Introduction

In this article, we consider the stochastic wave equation in spatial dimension $d = 1$, driven by the Lévy white noise L introduced in Balan (2015):

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + \sigma(u(t,x))\dot{L}(t,x) + b(u(t,x)), \quad t > 0, x \in \mathbb{R} \\ u(0,x) = v_0(x), \\ \dfrac{\partial u}{\partial t}(0,x) = v_1(x), \end{cases} \tag{1}$$

where v_0 is a bounded function and $v_1 \in L^1(\mathbb{R})$. We assume that σ and b are Lipschitz continuous functions. We let G be the fundamental solution of the wave equation on $\mathbb{R}$:

$$G(t,x) = \frac{1}{2} 1_{\{|x| \leq t\}}, \quad t > 0, x \in \mathbb{R}, \tag{2}$$

and w be the solution of the homogeneous wave equation on $\mathbb{R}$ with the same initial conditions as (1):

$$w(t,x) = \frac{1}{2}\int_{x-t}^{x+t} v_1(y)dy + \frac{1}{2}\Big(v_0(x+t) + v_0(x-t)\Big). \tag{3}$$

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We say that a predictable process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}\}$ is a (mild) *solution* of (1) if it satisfies the following integral equation:

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}} G(t-s, x-y)\sigma(u(s,y))L(ds,dy) + \int_0^t \int_{\mathbb{R}} G(t-s,x-y)b(u(s,y))dyds. \tag{4}$$

Before we proceed, we recall briefly from Balan (2015) the definition of the Lévy white noise L and the construction of the stochastic integral with respect to this noise.

We consider a Poisson random measure (PRM) N on the space $\mathbb{E} = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_0$, of intensity $\mu = dtdx\nu(dz)$, where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ and ν is a *Lévy measure* on $\mathbb{R}$, i.e.

$$\int_{\mathbb{R}_0} (1 \wedge |z|^2)\nu(dz) < \infty \quad \text{and} \quad \nu(\{0\}) = 0.$$

We denote by $\widehat{N}$ the compensated PRM defined by $\widehat{N}(A) = N(A) - \mu(A)$ for any Borel set A in $\mathbb{E}$ with $\mu(A) < \infty$. Throughout this article, we assume that ν satisfies:

$$m_2 := \int_{\mathbb{R}_0} |z|^2 \nu(dz) < \infty. \tag{5}$$

Suppose that N is defined on a complete probability space $(\Omega, \mathcal{F}, P)$. On this space, we consider the filtration

$$\mathcal{F}_t = \sigma(\{N([0,s] \times B \times \Gamma); 0 \leq s \leq t, B \in \mathcal{B}_b(\mathbb{R}), \Gamma \in \mathcal{B}_b(\mathbb{R}_0)\}) \vee \mathcal{N}, \quad t \geq 0,$$

where $\mathcal{N}$ is the class of P-negligible sets, $\mathcal{B}_b(\mathbb{R})$ is the class of bounded Borel sets in $\mathbb{R}$, and $\mathcal{B}_b(\mathbb{R}_0)$ is the class of Borel sets in $\mathbb{R}_0$ which are bounded away from 0. Similarly to Itô's classical theory, for any predictable process H which satisfies

$$E\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}_0} |H(s,x,z)|^2 \widehat{N}(ds,dx,dz) < \infty \quad \text{for all } t > 0, \tag{6}$$

we can define the stochastic integral of H with respect to $\widehat{N}$, and the integral process $\{\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}_0} H(s,x,z)\widehat{N}(ds,dx,dz); t \geq 0\}$ is a zero-mean square-integrable martingale (see for instance Section 2.2 of Kunita, 2004 or Section 4.2 of Applebaum, 2009). We work only with càdlàg modifications of such integral processes. (A process is *càdlàg* if its sample paths are right-continuous and have left limits.) Moreover, the following isometry property holds:

$$E\left|\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}_0} H(s,x,z)\widehat{N}(ds,dx,dz)\right|^2 = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}_0} |H(s,x,z)|^2 \nu(dz)dxds. \tag{7}$$

Here, we say that a process $H = \{H(t,x,z); t \geq 0, x \in \mathbb{R}, z \in \mathbb{R}_0\}$ is *predictable* if it is measurable with respect to the σ-field generated by all linear combinations of "elementary" processes, i.e. processes of the form $H(\omega,t,x,z) = Y(\omega)1_{(a,b]}(t)1_A(x)1_\Gamma(z)$, with $0 \leq a < b$, Y an $\mathcal{F}_a$-measurable bounded random variable, $A \in \mathcal{B}_b(\mathbb{R})$ and $\Gamma \in \mathcal{B}_b(\mathbb{R}_0)$.

The *Lévy white noise* defined in Balan (2015) is a "worthy" martingale measure $L = \{L_t(B); t \geq 0, B \in \mathcal{B}_b(\mathbb{R})\}$ in the sense of Walsh (1986), given by:

$$L_t(B) = \int_0^t \int_B \int_{\mathbb{R}_0} z\widehat{N}(ds,dx,dz).$$

This noise is characterized by the following properties: (i) $L_t(B_1), \ldots, L_t(B_k)$ are independent for any $t > 0$ and for any disjoint sets $B_1, \ldots, B_k \in \mathcal{B}_b(\mathbb{R})$; (ii) for any $0 \leq s < t$ and $B \in \mathcal{B}_b(\mathbb{R})$, $L_t(B) - L_s(B)$ is independent of $\mathcal{F}_s$ and has characteristic function:

$$E(e^{iu(L_t(B)-L_s(B))}) = \exp\left\{(t-s)|B|\int_{\mathbb{R}}(e^{iuz} - 1 - iuz)\nu(dz)\right\}, \quad u \in \mathbb{R},$$

where $|B|$ is the Lebesgue measure of B. Using Walsh' theory developed in Walsh (1986), for any predictable process $X = \{X(t,x); t \geq 0, x \in \mathbb{R}\}$ which satisfies the condition:

$$E\int_0^t \int_{\mathbb{R}} |X(s,x)|^2 dxds < \infty \quad \text{for any } t > 0, \tag{8}$$

we can define the stochastic integral of X with respect to L, and the integral process $\{\int_0^t \int_{\mathbb{R}} X(s,x)L(ds,dx); t \geq 0\}$ is a zero-mean square-integrable martingale. Moreover,

$$\int_0^t \int_{\mathbb{R}} X(s,x)L(ds,dx) = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}_0} X(s,x)z\widehat{N}(ds,dx,dz),$$