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Stokes' theorem, Stein's identity and completeness

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ABSTRACT

We study the relation between Stein's theorem and Stokes' theorem (or the divergence theorem) and show, using completeness of certain exponential families, that they are equivalent, in a certain sense, by using each to prove a version of the other.

1. Introduction

We study the relationships between Stein's lemma (Stein, 1981) and Stokes' theorem, and demonstrate using completeness of certain exponential families that they are in fact equivalent in a certain sense. Stein's lemma has been extensively used in the area of minimax shrinkage estimation in the normal case. Also, Stokes' theorem has been used to prove versions of Stein's lemma which are useful in developing minimax shrinkage procedures for general spherically and elliptically symmetric distributions.

In Section 2, we study the relation between the two results for spherical regions and prove, using Stein's lemma, an almost everywhere version of Stokes' theorem. Since Stein's lemma can be proved (e.g. Stein's original proof) without the use of Stokes' theorem, this implies a certain type of equivalence between the two results, as noted above.

In Section 3, we extend this equivalence to regions with quite general contours. We prove an extension of Stein's lemma to densities of the form $\exp(-\varphi(x))$ where φ is a smooth function tending to ∞ in all directions which defines level sets of the form $[\varphi \leq r]$ with boundaries of the form $[\varphi = r]$. The proof we give is analogous to Stein's original proof for the normal case and does not use Stokes' theorem. We also give an almost everywhere version of Stokes' theorem whose proof is based on the extended Stein's lemma and which again uses completeness of a certain exponential family in an essential way. For completeness, we give, in the Appendix, a proof of the extended version Stein's lemma using Stokes' theorem. This again establishes a certain equivalence between (generalized) Stein's lemma and Stokes' theorem for quite general regions. To us, it is quite striking that completeness can be used to establish a connection between these two well known results.

In Section 4, we give several examples, mostly known, of applications of Stein's lemma in estimation of a mean vector. Section 5 gives some concluding remarks.

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2. Equivalence of Stokes' theorem and Stein's lemma on spheres

Stein's lemma (Stein, 1981) and Stokes' theorem (see e.g. Stroock, 1990 and Kavian, 1990) are well known to statisticians, and indeed Stokes' theorem has been used to prove an extension of Stein's lemma to spherically symmetric distributions (including the normal) (see e.g. Fourdrinier and Strawderman, 2008). It is the purpose of this section to show that one may use Stein's lemma to derive a version of Stokes' theorem on spheres, so that, in a certain sense, the two results are equivalent. Note that, in Section 3, this connection will be extended to regions which are not spherical.

Here is Stein's lemma.

Lemma 2.1. *Stein's lemma (Stein, 1981). Let $X \sim \mathcal{N}_p(\theta, \tau^2 I)$ for fixed $\theta \in \mathbb{R}^p$ and $\tau > 0$. If g is a weakly differentiable function from $\mathbb{R}^p$ into $\mathbb{R}^p$ then, denoting by E_{θ,τ^2} the expectation with respect to $\mathcal{N}_p(\theta, \tau^2 I)$, we have*

$$E_{\theta,\tau^2}[(X-\theta)^t g(X)] = \tau^2 E_{\theta,\tau^2}[\mathrm{div} g(X)], \tag{2.1}$$

provided that either expectation exists.

Stein's lemma follows immediately from a generalization given in Section 3. For completeness, we provide two proofs of this generalization (see Lemma 3.1 in Section 3). The first proof, given in Section 3, follows along the lines of Stein's original proof and does not use Stokes' theorem. A second proof relying on Stokes' theorem is given in the Appendix.

The main result of this section proves an almost everywhere version of Stokes' theorem for weakly differentiable function from $\mathbb{R}^p$ into $\mathbb{R}^p$, when the sets of integration are balls $B_{r,\theta}$ and spheres $S_{r,\theta}$ of radius $r > 0$ and centered at $\theta \in \mathbb{R}^p$ and the measures are respectively the Lebesgue measure on $B_{r,\theta}$ and the uniform measure on $S_{r,\theta}$ (see Appendix A.1 for more details). The proof uses Stein's lemma and completeness of the normal scale family. Stokes' theorem follows as a corollary for functions which are sufficiently smooth near the inner boundary of $B_{r,\theta}$. Thus equivalence between Stein's lemma and Stokes' theorem is established for spheres and balls. As noticed above, in Section 3, we extend this equivalence to more general regions.

Theorem 2.1. *Let $r > 0$ and let $\theta \in \mathbb{R}^p$ be fixed. Let g be a weakly differentiable function from $\mathbb{R}^p$ into $\mathbb{R}^p$. Then, provided that the following integrals exist, we have, for almost every $r > 0$,*

$$\int_{S_{r,\theta}} \left(\frac{x-\theta}{\|x-\theta\|}\right)^t g(x)\, d\sigma_{r,\theta}(x) = \int_{B_{r,\theta}} \mathrm{div} g(x)\, dx, \tag{2.2}$$

where $\sigma_{r,\theta}$ is the uniform measure on $S_{r,\theta}$. Further, for every r for which

$$\lim_{r'\to r^-} \int_{S_{r',\theta}} \left(\frac{x-\theta}{\|x-\theta\|}\right)^t g(x)\, d\sigma_{r',\theta}(x) = \int_{S_{r,\theta}} \left(\frac{x-\theta}{\|x-\theta\|}\right)^t g(x)\, d\sigma_{r,\theta}(x) \tag{2.3}$$

the two integrals in (2.2) are equal.

Proof. Let $\theta \in \mathbb{R}^p$ and $\tau > 0$ be fixed and let $X \sim \mathcal{N}_p(\theta, \tau^2 I)$. Assume first that $E_{\theta,\tau^2}[|g(X)|] < \infty$ so that (2.1) holds. Integrating through uniform measures on spheres (equivalently through spherical coordinates), we have

$$\begin{aligned} E_{\theta,\tau^2}[(X-\theta)^t g(X)] &= \int_{\mathbb{R}^p} (x-\theta)^t g(x)\, \frac{1}{(2\pi\tau^2)^{p/2}} \exp\left(-\frac{\|x-\theta\|^2}{2\tau^2}\right) dx \\ &= \int_{\mathbb{R}_+} \int_{S_{r,\theta}} \left(\frac{x-\theta}{\|x-\theta\|}\right)^t g(x)\, d\sigma_{r,\theta}(x)\, \psi_{\tau^2}(r)\, dr \end{aligned} \tag{2.4}$$

where

$$\psi_{\tau^2}(r) = \frac{1}{(2\pi\tau^2)^{p/2}}\, r \exp\left(-\frac{r^2}{2\tau^2}\right). \tag{2.5}$$

Also

$$\begin{aligned} \tau^2 E_{\theta,\tau^2}[\mathrm{div} g(X)] &= \tau^2 \int_{\mathbb{R}^p} \mathrm{div} g(x)\, \frac{1}{(2\pi\tau^2)^{p/2}} \exp\left(-\frac{\|x-\theta\|^2}{2\tau^2}\right) dx \\ &= \int_{\mathbb{R}^p} \mathrm{div} g(x)\, \frac{1}{(2\pi\tau^2)^{p/2}} \left[-\tau^2 \exp\left(-\frac{r^2}{2\tau^2}\right)\right]_{\|x-\theta\|}^{\infty} dx \\ &= \int_{\mathbb{R}^p} \mathrm{div} g(x) \int_{\|x-\theta\|}^{\infty} \psi_{\tau^2}(r)\, dr\, dx \\ &= \int_{\mathbb{R}_+} \int_{B_{r,\theta}} \mathrm{div} g(x)\, dx\, \psi_{\tau^2}(r)\, dr, \end{aligned} \tag{2.6}$$