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Lamperti transformation for continuous-state branching processes with competition and applications

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ABSTRACT

The Lamperti transformation is established between continuous-state branching processes (CB-processes) with competition and strong solutions of a certain type of stochastic equations driven by Lévy processes without negative jumps. Using this result we study the maximal jumps of CB-processes with competition. In particular, we obtain the distributions of the maximal jumps of CB-processes and logistic branching processes.

1. Introduction

Continuous-state branching processes (CB-processes) were first introduced by Jiřina (1958). Lamperti (1967) provided a one-to-one correspondence between CB-processes and killed Lévy processes without negative jumps by random time changes. New proofs of the result were given by Caballero et al. (2009). Lambert (2005) defined the logistic branching processes (LB-processes) in terms of time-changed Ornstein–Uhlenbeck type processes to model competition within the population. More general competition models were introduced in Ba and Pardoux (2013, in press), Le (2014) and Berestycki et al. (2015).

In this work, we will introduce the CB-processes with more general competition in terms of a stochastic integral equation with jumps and a nonlinear drift. In Section 2 we show that these processes are in one-to-one correspondence with strong solutions to a certain type of stochastic differential equations driven by Lévy processes with no negative jumps. Finally in Section 3 we give an application of this result by studying the distribution of the maximal jump of a CB-process with competition.

2. Lamperti transformation for CB-processes with competition

In this section, we study the connection between Lévy processes with no negative jumps and CB-processes with competition. Let ϕ be a branching mechanism with the representation

$$\phi(\lambda) = b\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-z\lambda} - 1 + z\lambda)m(dz), \quad \lambda \geq 0, \tag{2.1}$$

where $b \in \mathbb{R}$ and $\sigma \geq 0$ are constants and $(z \wedge z^2)m(dz)$ is a finite measure on $(0, \infty)$.

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Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ be a complete filtered probability space. Let $\{B_t\}$ be a standard Brownian motion and $\tilde{N}(ds, dz, du)$ a compensated Poisson random measure on $(0, \infty)^3$ with intensity $dsm(dz)du$. Let β be a continuous non-decreasing function on $[0, \infty)$ with $\beta(0) = 0$. Given $Y_0 > 0$, consider the following stochastic equation

$$Y_t = Y_0 - b\int_0^t Y_s ds - \int_0^t \beta(Y_s)ds + \sigma\int_0^t \sqrt{Y_s}dB_s + \int_0^t\int_0^\infty\int_0^{Y_{s-}} z\tilde{N}(ds, dz, du). \tag{2.2}$$

We understand the last term on the right hand side as an integral over the set $\{(s, z, u) : 0 < s \le t, 0 < z < \infty, 0 < u \le Y_{s-}\}$ and give similar interpretations for other integrals with respect to Poisson random measures in this article. By a *solution* of the stochastic equation to (2.2), we understand a càdlàg and $(\mathscr{F}_t)$-adapted process $\{Y_t : t \ge 0\}$ that satisfies Eq. (2.2) almost surely for every $t \ge 0$. We say $\{Y_t : t \ge 0\}$ is a *strong solution* if, in addition, it is adapted to the augmented natural filtration generated by $\{B_t\}$ and $\{N(ds, dz, du)\}$.

By Proposition 5.1 in Fu and Li (2010), there is a unique non-negative strong solution $\{Y_t : t \ge 0\}$ to (2.2); see also Dawson and Li (2006, 2012). Note that $\{Y_t : t \ge 0\}$ is a CB-process with branching mechanism ϕ if $\beta \equiv 0$. We call $\{Y_t : t \ge 0\}$ a CB-process with competition, with branching mechanism ϕ and competition mechanism β. Some special cases of this model have been studied by a number of authors; see, e.g., Ba and Pardoux (in press), Lambert (2005), Le (2014), Le and Pardoux (in press) and Le et al. (2013).

Clearly, zero is a trap for $\{Y_t : t \ge 0\}$. Define $T_0 = \sup\{t : Y_t > 0\}$. The functional

$$t \mapsto \eta(t) = \int_0^t Y_s ds \tag{2.3}$$

is then continuous and strictly increasing as long as $t \in [0, T_0)$. Let $K = \eta(T_0) \equiv \eta(\infty)$ and $t \mapsto C(t)$ be the right inverse of η. Then $C(t)$ is a continuous strictly increasing function with values in $[0, T_0)$ for $t \in [0, K)$, and $C(t) = \infty$ for $t \in [K, \infty)$. Moreover, we clearly have

$$C(\eta(s)) = s \quad \text{for } s \in [0, T_0), \qquad \eta(C(t)) = t \quad \text{for } t \in [0, K).$$

Theorem 2.1. *Define* $\{R_t\}$ *by*

$$R_t = Y_{C(t)\wedge T_0}, \quad t \ge 0. \tag{2.4}$$

Then $\{R_t\}$ *is a solution of the following SDE:*

$$dR_t = 1_{\{R_{r-}>0:r\le t\}}dL_t - 1_{\{R_{r-}>0:r\le t\}}\frac{\beta(R_t)}{R_t}dt, \qquad R_0 = Y_0, \tag{2.5}$$

where $\{L_t\}$ *is a Lévy process generated by* $(\sigma^2, m, -b)_1$*; see, e.g.,* Sato (1999, p. 65).

Remark 1. Since $\{Y_t : t \ge 0\}$ has no negative jumps and $Y_{t-} = 0$ implies $Y_t = 0$, we get $Y_{t-} > 0$ iff $Y_t > 0$ iff $t \in [0, T_0)$. Thus $R_{t-} > 0$ iff $R_t > 0$ iff $t \in [0, K)$. Then for any $t \in [0, K)$, (2.5) is equivalent to

$$R_t = Y_0 + L_t - \int_0^t \frac{\beta(R_s)}{R_s}ds. \tag{2.6}$$

Proof. By (2.2) and (2.4), we have

$$\begin{aligned} R_t &= Y_0 - b\int_0^{C(t)\wedge T_0} Y_s ds - \int_0^{C(t)\wedge T_0}\beta(Y_s)ds + \sigma\int_0^{C(t)\wedge T_0}\sqrt{Y_s}dB_s \\ &\quad + \int_0^{C(t)\wedge T_0}\int_0^\infty\int_0^{Y_{s-}} z\tilde{N}(ds, dz, du), \quad t \ge 0. \end{aligned} \tag{2.7}$$

It is elementary to see that

$$t \mapsto \int_0^{C(t)\wedge T_0}\sqrt{Y_s}dB_s$$

is a continuous local martingale with increasing process

$$\int_0^{C(t)\wedge T_0} Y_s ds = t \wedge K.$$

Thus, there exists a Brownian motion $\{\tilde{B}_t\}$ such that

$$\int_0^{C(t)\wedge T_0}\sqrt{Y_s}dB_s = \tilde{B}_{t\wedge K}. \tag{2.8}$$