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On the non-existence of maximum likelihood estimates for the extended exponential power distribution and its generalizations

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ABSTRACT

In this work, we show that the maximum likelihood estimates fail to exist for the extended exponential power distribution and its univariate generalizations.

1. Introduction

The probability density function of the extended exponential power distribution (EEPD) is

$$f(x|\mu,\sigma,\alpha)=\frac{\alpha}{2\sigma\Gamma(1/\alpha)}\exp\left\{-\left(\frac{|x-\mu|}{\sigma}\right)^{\alpha}\right\},\quad x\in\mathcal{R},$$

where $\mu \in \mathcal{R}$; $\sigma, \alpha \in \mathcal{R}^+$. This distribution is a three-parameter symmetric distribution that includes the Laplace and normal distributions as special cases. The standard EEPD was introduced by Subbotin (1923) as a "Generalized Law of Errors". Choy and Walker (2003) used the name "Extended Exponential Power Distribution" to emphasize the fact that the EEPD extends the parameter space of the exponential power distribution (Box and Tiao, 1973) so that the shape parameter can be any positive real number. Despite the need for this distinction, the EEPD is often referred to simply as the exponential power distribution. The EEPD has also been referred to as Subbotin's distribution and the generalized normal distribution (Nadarajah, 2005); reparameterizations of the EEPD include the generalized Gaussian distribution (Beaulieu and Guo, 2012) and the normal distribution of order p (Mineo, 2003). Univariate generalizations of the EEPD include the type I and type II asymmetric Subbotin distributions (Azzalini and Capitanio, 2014), the four-parameter asymmetric exponential power distribution (AEPD) (Ayebo and Kozubowski, 2003), and the five-parameter AEPDs of Bottazzi and Secchi (2011) and Zhu and Zinde-Walsh (2009). The EEPD has also been used as an error distribution in the linear regression model (Zeckhauser and Thompson, 1970).

The problem of maximum likelihood estimation for these distributions has been considered by several authors. Agrò (1995) considered numerical maximum likelihood estimation for the normal distribution of order p. Note that this author refers to this distribution as the exponential power function. Bottazzi and Secchi (2011) and Zhu and Zinde-Walsh (2009) considered numerical maximum likelihood estimation for their respective versions of the five-parameter AEPD and

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Zeckhauser and Thompson (1970) used numerical maximum likelihood estimation to estimate the parameters of the simple linear regression model with EEPD errors. Based on their numerical studies, Zhu and Zinde-Walsh (2009) pointed out that the maximum likelihood estimates may not exist for their version of the five-parameter AEPD when the sample size is less than 500; Agrò (1995) made a similar observation for the normal distribution of order p based on numerical results. The main results of this work show that the maximum likelihood estimates always fail to exist for the EEPD (or any invertible reparameterization of the EEPD) and its univariate generalizations. This includes all the reparameterizations and univariate generalizations of the EEPD discussed so far. We prove the main results in Section 2 and give concluding remarks in Section 3.

2. Main results

The following theorem establishes the fact that the likelihood function for the EEPD is unbounded for any sample size, and thus its maximum likelihood estimates always fail to exist.

Theorem 1. *Let $L(\mu, \sigma, \alpha|\boldsymbol{x})$ denote the likelihood function for an independent and identically distributed (i.i.d.) sample of size $n \geq 1$ drawn from the EEP (μ, σ, α) distribution. Then, $\lim_{\sigma\to 0^+} L(x_1, \sigma, 1|\boldsymbol{x}) \to \infty$ when $n = 1$ or $x_1 = x_2 = \cdots = x_n$ and $\lim_{\alpha\to 0^+} L(x_1, \sigma^*, \alpha|\boldsymbol{x}) \to \infty$ when $n > 1$ and $x_i \neq x_j$ for some $i \neq j$, where $\sigma^* = \left(\frac{\alpha}{n}\sum_{i=2}^{n}|x_i - x_1|^\alpha\right)^{1/\alpha}$.*

Proof. The likelihood function can be written as

$$L(\mu, \sigma, \alpha|\boldsymbol{x}) = \left(\frac{\alpha}{2\sigma\Gamma(1/\alpha)}\right)^n \exp\left\{-\sum_{i=1}^{n}\left(\frac{|x_i - \mu|}{\sigma}\right)^\alpha\right\}, \quad \mu \in \mathcal{R};\ \sigma, \alpha \in \mathcal{R}^+,$$

where $\boldsymbol{x} \in \mathcal{R}^n$. When $n = 1$ or $x_1 = x_2 = \cdots = x_n$, we have $\lim_{\sigma\to 0^+} L(x_1, \sigma, 1|\boldsymbol{x}) = \lim_{\sigma\to 0^+}\left(\frac{1}{2\sigma}\right)^n \to \infty$. Now, consider the case of $n > 1$ and $x_i \neq x_j$ for some $i \neq j$. We can write

$$L\left(x_1, \sigma^*, \alpha|\boldsymbol{x}\right) = \left(\frac{\alpha^{1-(1/\alpha)}}{2\Gamma(1/\alpha)}\left(\frac{n}{\sum_{i=2}^{n}|x_i - x_1|^\alpha}\right)^{1/\alpha}\right)^n \exp\left(-\frac{n}{\alpha}\right).$$

To facilitate the use of asymptotic results, we express $\lim_{\alpha\to 0^+} L(x_1, \sigma^*, \alpha|\boldsymbol{x})$ as

$$\lim_{\alpha\to\infty}\left(\frac{(1/\alpha)^{1-\alpha}}{2\Gamma(\alpha)}\left(\frac{n}{\sum_{i=2}^{n}|x_i - x_1|^{1/\alpha}}\right)^\alpha\right)^n \exp(-n\alpha). \tag{1}$$

Using the fact $\Gamma(\alpha) \sim \sqrt{2\pi/\alpha}\,(\alpha/e)^\alpha$ (Abramowitz and Stegun, 1986, p. 257), we determine that the expression to the right of the limit sign in (1) is asymptotically equivalent to

$$\left(\frac{1}{2\sqrt{2\pi\alpha}}\left(\frac{n}{\sum_{i=2}^{n}|x_i - x_1|^{1/\alpha}}\right)^\alpha\right)^n.$$

The limit of this expression can be written as

$$\lim_{\alpha\to\infty}\left(\frac{(n/(n-1))^\alpha}{2\sqrt{2\pi\alpha}}\left(\frac{n-1}{\sum_{i=2}^{n}|x_i - x_1|^{1/\alpha}}\right)^\alpha\right)^n. \tag{2}$$

Consider the limit of the first factor in the first set of parentheses. Since the numerator and denominator of this factor both have a limit of ∞, we can use L'Hopital's rule to write

$$\lim_{\alpha\to\infty}\frac{(n/(n-1))^\alpha}{2\sqrt{2\pi\alpha}} = \lim_{\alpha\to\infty}\log\left(\frac{n}{n-1}\right)\left(\frac{n}{n-1}\right)^\alpha\sqrt{\frac{\alpha}{2\pi}} \to \infty.$$

The limit of the second factor in the first set of parentheses in (2) can be written as

$$\lim_{\alpha\to\infty}\exp\left\{\frac{\log(n-1) - \log\left(\sum_{i=2}^{n}|x_i - x_1|^{1/\alpha}\right)}{1/\alpha}\right\}. \tag{3}$$