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# Translation, contraction and dilation of dual generalized order statistics

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ABSTRACT

A general class of distributions has been characterized through translation of two non-adjacent dual generalized order statistics (dgos). Moreover, the characterizing results are obtained for generalized Pareto distribution through dilation of dgos and generalized power function distribution through contraction of non-adjacent generalized order statistics.



## 1. Introduction

Generalized order statistics (gos) have been introduced by Kamps (1995) as a unification of several models of ascendingly ordered random variables (rv's). The gos $X(1,n,\tilde{m},k),\ X(2,n,\tilde{m},k),\ \ldots, X(n,n,\tilde{m},k)$ based on a distribution function (df) $F$ are defined by their probability density function (pdf)

$$f^{(\tilde{m},k)}_{1,2,\ldots,n:n}(x_1,\ldots,x_n) = k\left(\prod_{i=1}^{n-1}\gamma_i^{(n)}\right)\left(\prod_{i=1}^{n-1}\overline{F}^{m_i}(x_i)\right)\overline{F}^{k-1}(x_n)\left(\prod_{i=1}^{n}f(x_i)\right),$$

on the cone $\{(x_1,\ldots,x_n) : F^{-1}(0) < x_1 \le \cdots \le x_n < F^{-1}(1)\}$, where $\overline{F} = 1-F$. The parameters $\gamma_1^{(n)},\ldots,\gamma_n^{(n)}$ are defined by $\gamma_n^{(n)} = k > 0$ and $\gamma_r^{(n)} = k+n-r+\sum_{j=r}^{n-1} m_j > 0,\ r = 1,2,\ldots,n-1$, where $\tilde{m} = (m_1,\ldots,m_{n-1}) \in \Re^{n-1}$. Particular choices of the parameters $\gamma_1^{(n)},\ldots,\gamma_n^{(n)}$ lead to different models, e.g., ordinary order statistics ($m_1 = m_2 = \cdots = m_{n-1} = 0, k = 1$); order statistics with non-integral sample size ($m_1 = m_2 = \cdots = m_{n-1} = 0, k = \alpha - n + 1$, and $\alpha > n-1$); $k$th record values ($m_1 = m_2 = \cdots = m_{n-1} = -1$ and $k$ is any positive integer) and sequential order statistics ($m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1} - 1, 1 \le i \le n-1, k = \alpha_n$ and $\alpha_1,\alpha_2,\ldots,\alpha_n > 0$).

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The concept of dgos is introduced in Burkschat et al. (2003) to enable a common approach to descendingly ordered rv's like reversed order statistics and lower records models. The dgos $X^*(1, n, \tilde{m}, k), X^*(2, n, \tilde{m}, k), \dots, X^*(n, n, \tilde{m}, k)$ based on a df $F$ are defined by their pdf

$$f^{*(\tilde{m},k)}_{1,2,\dots,n.n}(x_1, \dots, x_n) = k\left(\prod_{i=1}^{n-1}\gamma_i^{(n)}\right)\left(\prod_{i=1}^{n-1}F^{m_i}(x_i)\right)(F^{k-1}(x_n))\left(\prod_{i=1}^{n}f(x_i)\right),$$

where $F^{-1}(1) > x_1 \geq \dots \geq x_n > F^{-1}(0)$.

In this work, we consider a wide subclass of gos (dgos), by assuming $m_1 = m_2 = \dots = m_{n-1} = m$. This subclass is known as $m$-gos ($m$-dgos). Clearly, the class of $m$-gos includes many important particular models such as ordinary order statistics, order statistics with non-integer sample size, record values and sequential order statistics. The marginal df's of the $r$th $m$-gos $X(r, n, m, k)$ and $m$-dgos $X^*(r, n, m, k)$ are given, respectively, by

$$f_{X(r,n,m,k)} = \frac{C^n_{r-1}}{(r-1)!}\overline{F}^{\gamma_r^{(n)}-1}(x)\left[\frac{1-\overline{F}^{m+1}(x)}{m+1}\right]^{r-1} f(x), \quad m \neq -1, \tag{1.1}$$

and

$$f_{X^*(r,n,m,k)} = \frac{C^n_{r-1}}{(r-1)!}F^{\gamma_r^{(n)}-1}(x)\left[\frac{1-F^{m+1}(x)}{m+1}\right]^{r-1} f(x), \quad m \neq -1, \tag{1.2}$$

where (1.2) is obtained just by replacing $\overline{F}$ by $F$, $\gamma_r^{(n)} = k + (n-r)(m+1)$, $1 \leq r \leq n$, and $C^n_{r-1} = \prod_{i=1}^{r}\gamma_i^{(n)}$, $1 \leq r \leq n$.

Ahsanullah (2004) has characterized uniform distribution under random contraction for adjacent dgos. In this paper, distributional properties of the dgos have been used to characterize a general form of distributions for non-adjacent dgos under random translation, dilation and contraction, thus generalizing the results of Ahsanullah (2004) and Beutner and Kamps (2008). Furthermore, results in terms of gos and ordinary order statistics are deduced. One may also refer to Wesolowski and Ahsanullah (2004), Arnold et al. (2008), Castaño-Martínez et al. (2012) and Khan and Shah Imtiyaz (2012) for the related results. The following two elementary lemmas will be needed in our study.

**Lemma 1.1.** *Let $h$ be a measurable function of the rv $X$, i.e., $Y = h(X)$ is a rv. Then*

1. *$Y(r, n, m, k) = h(X(r, n, m, k))$ and $Y^\star(r, n, m, k) = h(X^\star(r, n, m, k))$, if $h$ is an increasing function (e.g., $Y_{r:n} = h(X_{r:n})$ where $X_{r:n}$ is the $r$th order statistic).*
2. *$Y(r, n, m, k) = h(X^\star(r, n, m, k))$ and $Y^\star(r, n, m, k) = h(X(r, n, m, k))$, if $h$ is a decreasing function (e.g., $Y_{n-r+1:n} = h(X_{r:n})$).*

**Lemma 1.2.** 1. *if $Y = \log X \sim exp(\frac{\alpha}{m+1})$, $m > -1, \alpha > 0$ (exponential df, i.e., $F_Y(y) = 1 - e^{-\frac{\alpha y}{m+1}}$, $0 < y < \infty, \alpha > 0, m > -1$), then $X \sim Par(\frac{\alpha}{m+1})$ (Pareto df, i.e., $F_X(x) = 1 - x^{-\frac{\alpha}{m+1}}$, $1 < x < \infty$).*
2. *if $-\log X \sim exp(\frac{\alpha}{m+1})$, $m > -1, \alpha > 0$, then $X \sim pow(\frac{\alpha}{m+1})$ (power function df, i.e., $F_X(x) = x^{\frac{\alpha}{m+1}}$, $0 < x < 1$).*
3. *if $Y = \log X \sim genexp(\alpha)$ (i.e., $F_Y(y) = [1-(m+1)e^{-\alpha y}]^{\frac{1}{m+1}}$, $\frac{1}{\alpha}\log(m+1) < y < \infty, \alpha > 0, m > -1$), then $X \sim genPar(\alpha)$ (i.e., $F_x(x) = [1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}$, $(m+1)^{\frac{1}{\alpha}} < x < \infty, \alpha > 0$).*
4. *if $-\log X \sim genexp(\alpha)$, then $X \sim genpow(\alpha)$ (i.e., $F_X(x) = 1 - [1-(m+1)x^{\alpha}]^{\frac{1}{m+1}}$, $0 < x < (m+1)^{\frac{1}{\alpha}}$).*

## 2. Characterization results based on gos and dgos

We assume that all the considered df's are differentiable with respect to their arguments.

**Theorem 2.1.** *Let $X^\star(r, n, m, k)$ be the $r$th dgos from a sample of size $n$ drawn from a continuous population with the pdf $f_X$ and the df $F_X$. Furthermore, let $Y_j$, $j = 0, 1, \dots, n_1 - n_2 - 1$, be rv's, which are independent of $X$ and satisfy the relation*

$$X^\star(r, n_1 - j, m, k) \overset{d}{=} X^\star(r, n_2, m, k) + Y_j, \quad 1 \leq r < n_2 < n_1,\ j = 0, 1, \dots, n_1 - n_2 - 1, \tag{2.1}$$

*where "$X \overset{d}{=} Y$" means that the rv's $X$ and $Y$ have the same df's. Then $Y_j \overset{d}{=} Y(n_1 - n_2 - j; n_1 - j, m, k)$, where $Y(n_1 - n_2 - j; n_1 - j, m, k)$ is the $(n_1 - n_2 - j)$th $m$-gos from a sample of size $(n_1 - j)$ drawn from a df $F_Y$ and $Y \sim exp(\frac{\alpha}{m+1})$ if and only if $X \sim genexp(\alpha)$.*

**Proof.** We first prove the necessity part. Let the moment generating function (mgf) of $X^\star(r, n_1, m, k)$ be $M_{X^\star(r,n_1,m,k)}(t)$. Then, (2.1) implies that

$$M_{X^\star(r,n_1-j,m,k)}(t) = M_{X^\star(r,n_2,m,k)}(t)M_{Y_j}(t).$$