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Limiting behaviour of constrained sums of two variables and the principle of a single big jump

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ABSTRACT

This note studies the asymptotic properties of the variable $Z_d := \frac{X_1}{d}\Big|\{X_1 + X_2 = d\}$, as $d \to \infty$. Here X_1 and X_2 are non-negative i.i.d. variables with a common twice differentiable density function f. General results concerning the distributional limits of Z_d are discussed with various examples.

1. Preliminaries

A cornerstone of heavy-tailed thinking is the *principle of a single big jump*. Unfortunately, there does not seem to exist consensus about the exact definition of this principle. Nevertheless, the principle always consists of the idea that the most likely way for a sum to be large is that one of the summands is large. Some authors refer to this principle whenever there exists a dominating random variable (Embrechts et al., 1997; Foss et al., 2007), whereas others reserve the expression for subexponential distributions (Armendáriz and Loulakis, 2011; Asmussen and Klüppelberg, 1996; Denisov et al., 2008) or their generalisations (Beck et al., in press). Some properties are also studied in the case of dependent variables (Albrecher et al., 2006).

In Foss et al. (2013), the behaviour of the process $(Z_d) := (Z_d)_{d>0}$, where

$$Z_d := \frac{X_1}{d}\Big|\{X_1 + X_2 = d\},$$

is used to illustrate the phenomenon of a single big jump. Our plan is to study the process (Z_d) further and to present general results whose applicability can be verified using the density function f.

In order to do this, we define two convergence types for the process (Z_d):

(I) $\mathcal{L}(Z_d) \to \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ and
(II) $\mathcal{L}(Z_d) \to \delta_{\frac{1}{2}}$.

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In I and II the notation $\mathcal{L}(Z_d)$ refers to the law of Z_d and the convergence is understood as convergence in distribution in the limit $d \to \infty$. In Types I and II, δ_x signifies a distribution concentrated to the point $x \in \{0, 1/2, 1\}$.

Behaviour I resembles the way many heavy-tailed variables are known to behave: if the sum $X_1 + X_2$ is large then one of the variables is large. Behaviour II is related to a phenomenon encountered within the class of light-tailed distributions: both of the variables X_1 and X_2 contribute equally.

Recall that a non-negative random variable X is called *heavy-tailed* if $E(e^{sX}) = \infty$ for all $s > 0$ and light-tailed otherwise. We will show that, in the sense of Behaviour I, the principle can occur outside the class of heavy-tailed distributions. Similar result has been obtained in Beck et al. (in press) in their framework. Traditionally the idea of the principle of a single big jump is almost exclusively associated with a subclass of heavy-tailed distributions called subexponential distributions. The subexponential class and its extensions are further discussed in Section 3.

1.1. Assumptions

The non-negative random variables X_1 and X_2 are independent and identically distributed. The variable X_1 has an unbounded support and a density function f. Set $F(x) := P(X_1 \leq x)$ and $\overline{F}(x) := 1 - F(x)$. The function f is assumed to be twice differentiable in the set $[0, \infty)$ and eventually decreasing. A property is said to hold *eventually* if there exists $y_0 \in \mathbb{R}$ such that the property is valid in the set $[y_0, \infty)$.

1.2. Basic properties

The density function f_{Z_d} of the variable Z_d is concentrated in the interval [0,1] and given by formula

$$f_{Z_d}(x) = \frac{f(dx)f(d(1-x))}{\int_0^1 f(dy)f(d(1-y))\,dy}, \quad x \in [0, 1]. \tag{1.1}$$

It can be directly obtained from the conditional distribution of X_1/d given $X_1 + X_2$ by a transformation of variables in the resulting integral. Formula (1.1) may not be defined for all d. It is well-defined eventually, as d grows. This poses no limitations to the following analysis, because the results concern the asymptotics of Z_d in the limit $d \to \infty$. The function f_{Z_d} can be viewed as a function of two variables as $g(x, d) := f_{Z_d}(x): [0, 1] \times (d_0, \infty) \to [0, \infty)$ for a suitable $d_0 > 0$. For a fixed d the function $f_{Z_d}(x)$ is symmetric with respect to the point $x = 1/2$. Hence, it suffices to formulate the results only for $x \in [0, 1/2]$.

Conditions implying Behaviour I or II typically involve estimation of decay rates of integrals. What is more, neither of the behaviours needs to occur; the distributional limit may exist without any concentration of probability mass. To see this, consider the following example.

Example 1.1. Suppose f is a gamma density function $f(x) = Cx^{a-1}e^{-x}$, where $x > 0$, $a > 0$ and $C > 0$ is an integration constant. Then f_{Z_d} of (1.1) reduces to $f_{Z_d}(x) = x^{a-1}(1-x)^{a-1}/\int_0^1 y^{a-1}(1-y)^{a-1}\,dy$, for all $d > 0$. So, $\mathcal{L}(Z_d)$ does not depend on d and belongs to the family of Beta distributions.

In order to understand the behaviour of the process (Z_d) one needs additional assumptions to those made in Section 1.1. One way to proceed is to demand that the function f_{Z_d} should eventually stay convex or concave at the midpoint of [0,1]. This leads to the following characterisation.

Lemma 1.2. *Suppose*

$$L := \lim_{x\to\infty} \operatorname{sign}\left(\frac{d^2}{dx^2}\log f(x)\right) \tag{1.2}$$

exists, where

$$\operatorname{sign}(x) := \begin{cases} 1 & : x > 0 \\ 0 & : x = 0 \\ -1 & : x < 0. \end{cases}$$

Then the function f_{Z_d} of Formula (1.1) is eventually, in d, strictly convex with respect to the variable x at point $x = 1/2$ if and only if $L = 1$. Similarly, f_{Z_d} is eventually, in d, strictly concave with respect to the variable x at point $x = 1/2$ if and only if $L = -1$.

Proof. Consider the eventually convex case; the eventually concave case is analogous. Let $d > 0$. For any $x \in (0, 1)$,

$$f''_{Z_d}(x) = \frac{d^2}{\int_0^1 f(dy)f(d(1-y))\,dy}[f''(dx)f(d(1-x)) - f'(dx)f'(d(1-x)) - f'(dx)f'(d(1-x)) + f(dx)f''(d(1-x))].$$