Statistics and Probability Letters 107 (2015) 199–203

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

When are two Markov chains similar?

Brian Fralix

Department of Mathematical Sciences, Clemson University, Clemson, SC, USA

ARTICLE INFO

Article history:
Received 23 July 2015
Received in revised form 25 August 2015
Accepted 28 August 2015
Available online 7 September 2015

MSC:
60J27
60J35

Keywords:
Similar Markov chain
Transient behavior
Transition function

ABSTRACT

We present sufficient conditions for two Markov chains to be similar, thus confirming a conjecture of Pollett (2001). Our proof technique also shows one of the sufficient conditions given in Theorem 2 of Lenin et al. (2000) for two birth–death processes to be similar is unnecessary.

1. Introduction

This note addresses the question of when two continuous-time Markov chains (CTMCs) are similar, as defined in Lenin et al. (2000) and Pollett (2001).

Suppose $X := \{X(t); t \geq 0\}$ and $\overline{X} := \{\overline{X}(t); t \geq 0\}$ are two CTMCs having state space E, and rate matrices $\mathbf{Q} := [q(x, y)]_{x,y\in E}$ and $\overline{\mathbf{Q}} := [\overline{q}(x, y)]_{x,y\in E}$, respectively. Further associated with X is its collection of transition times $\{T_n\}_{n\geq 0}$, where $T_0 := 0$, as well as its embedded discrete-time Markov chain (DTMC) $\{X_n\}_{n\geq 0}$, where $X_n := X(T_n)$ for each integer $n \geq 0$. Similarly, we let $\{\overline{T}_n\}_{n\geq 0}$ and $\{\overline{X}_n\}_{n\geq 0}$ represent the transition times and the embedded DTMC associated with $\overline{X}$.

We assume X and $\overline{X}$ share the same transition structure, meaning for any pair of states $x, y \in E$, $x \neq y$, $q(x, y) > 0$ if and only if $\overline{q}(x, y) > 0$. This assumption ensures both chains share the same communicating classes. Furthermore, we do not force either X or $\overline{X}$ to be regular, meaning either chain could explode in a finite amount of time: to model such a phenomenon, we create an additional state Δ outside of E, we let $T := \lim_{n\to\infty} T_n$ represent the time at which X explodes, and we define

$$X(t) = \begin{cases} \sum_{n=0}^{\infty} X_n \mathbf{1}(T_n \leq t < T_{n+1}), & t < T; \\ \Delta, & t \geq T \end{cases}$$

with $\overline{X}$ being constructed analogously, so that both processes are well-defined on $[0, \infty)$. Finally, we assume both X and $\overline{X}$ are stable and conservative over E, meaning $\mathbf{Q}$ satisfies $q(x) := -q(x, x) = \sum_{y\neq x} q(x, y) < \infty$ for each $x \in E$, and similarly for $\overline{\mathbf{Q}}$.

E-mail address: bfralix@clemson.edu.

http://dx.doi.org/10.1016/j.spl.2015.08.029

It is known – see e.g. Section 2.2 of Anderson (1991) – that the transition functions $\{p_{x,y}\}_{x,y\in E}$ associated with X are the minimal transition functions that solve the corresponding Kolmogorov forward and backward equations, where for each $t \geq 0, x, y \in E$,

$$p_{x,y}(t) := \mathbb{P}_x(X(t) = y)$$

with $\mathbb{P}_x$ representing a conditional probability, given $X(0) = x$. Throughout we will still use the notation $\mathbb{P}_x$ when conditioning on other Markov chains being in state x at time zero, with $\mathbb{E}_x$ being the corresponding expectation: this usage of notation is fairly standard, and should not cause any additional confusion. For example, for each $x, y \in E$ we follow the convention that the transition function $\overline{p}_{x,y}$ associated with $\overline{X}$ evaluated at time t, $\overline{p}_{x,y}(t)$, can also be expressed as $\mathbb{P}_x(\overline{X}(t) = y)$.

Our goal is to find sufficient conditions for X and $\overline{X}$ to be similar over a common communicating class C, where similarity is defined as follows.

Definition 1.1. We say X and $\overline{X}$ are similar over a communicating class C if there exists a collection of positive numbers $\{c_{x,y}\}_{x,y\in C}$ satisfying, for each $x, y \in C$,

$$\overline{p}_{x,y}(t) = c_{x,y}p_{x,y}(t), \quad t \geq 0.$$

The next theorem is the main result of this note: it will be proven in Section 3.

Theorem 1.1. *Suppose that for some communicating class C of both X and $\overline{X}$, the following conditions are satisfied:*

(a) *for each $x \in C$, $q(x) = \overline{q}(x)$.*
(b) *for each $x \in C$, every feasible path $x = x_0, x_1, \ldots, x_{n-1}, x_n = x$ under $\mathbf{Q}$ (and also $\overline{\mathbf{Q}}$) starting and ending at x satisfies*

$$\prod_{\ell=1}^{n} q(x_{\ell-1}, x_\ell) = \prod_{\ell=1}^{n} \overline{q}(x_{\ell-1}, x_\ell).$$

Then X and $\overline{X}$ are similar over C.

The next lemma provides a simple, yet important consequence of Conditions (a) and (b) and will be used in Section 3 to prove Theorem 1.1.

Lemma 1.1. *Suppose Condition* (b) *of Theorem* 1.1 *holds, and let x, y be two distinct states in C. Then for any two feasible paths $x = x_0, x_1, \ldots, x_{m-1}, x_m = y$ and $x = y_0, y_1, \ldots, y_{n-1}, y_n = y$ under $\mathbf{Q}$ from x to y, we have*

$$\prod_{\ell=1}^{m} \frac{\overline{q}(x_{\ell-1}, x_\ell)}{q(x_{\ell-1}, x_\ell)} = \prod_{\ell=1}^{n} \frac{\overline{q}(y_{\ell-1}, y_\ell)}{q(y_{\ell-1}, y_\ell)}.$$

Proof. Observe first that since $x, y \in C$, there exists a feasible path $y = z_0, z_1, \ldots, z_{k-1}, z_k = x$ from y to x. Furthermore, Condition (b) also implies

$$\prod_{\ell=1}^{k} \frac{\overline{q}(z_{\ell-1}, z_\ell)}{q(z_{\ell-1}, z_\ell)} \prod_{\ell=1}^{m} \frac{\overline{q}(x_{\ell-1}, x_\ell)}{q(x_{\ell-1}, x_\ell)} = 1 = \prod_{\ell=1}^{k} \frac{\overline{q}(z_{\ell-1}, z_\ell)}{q(z_{\ell-1}, z_\ell)} \prod_{\ell=1}^{n} \frac{\overline{q}(y_{\ell-1}, y_\ell)}{q(y_{\ell-1}, y_\ell)} \tag{1}$$

since $z_0, z_1, \ldots, z_{k-1}, z_k = x_0, x_1, \ldots, x_m$ and $z_0, z_1, \ldots, z_{k-1}, z_k = y_0, y_1, \ldots, y_n$ are both feasible paths starting and ending at state y. After dividing the far-left and far-right terms of (1) by their common multiple, we find

$$\prod_{\ell=1}^{m} \frac{\overline{q}(x_{\ell-1}, x_\ell)}{q(x_{\ell-1}, x_\ell)} = \prod_{\ell=1}^{n} \frac{\overline{q}(y_{\ell-1}, y_\ell)}{q(y_{\ell-1}, y_\ell)}$$

which proves our claim. ◇

Conditions (a) and (b) from Theorem 1.1 were conjectured on p. 64 of Pollett (2001) to be sufficient for two Markov chains to be similar over C. Theorem 1.1 also generalizes Theorem 2 of Lenin et al. (2000) where two sufficient conditions equivalent to (a) and (b) are given for two birth–death processes to be similar. In fact (as also pointed out in Pollett, 2001) the authors of Lenin et al. (2000) further assume an associated Stieltjes moment problem has a unique solution, but the arguments we provide here show this extra assumption is not needed to establish similarity, once Conditions (a) and (b) are known to hold.

Interestingly, in Lenin et al. (2000) the authors also show (see Theorem 1 of Lenin et al., 2000) that similarity of two birth–death processes over a communicating class C implies Conditions (a) and (b) of Theorem 1.1 must hold on C, i.e. that these are also necessary conditions for similarity on C. The author has tried showing that – as also conjectured in Pollett (2001) – Conditions (a) and (b) are also necessary for similarity to hold over C in this more general context, but currently it is not clear this is the case.