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On the weak laws of large numbers for sums of negatively associated random vectors in Hilbert spaces

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ABSTRACT

This note establishes the weak law of large numbers for sums of negatively associated random vectors in Hilbert spaces. Illustrative examples are provided, one of which shows that the conditions for the weak law of large numbers cannot be dispensed with.

1. Introduction

A finite sequence $\{X_1, \ldots, X_n\}$ of random variables is said to be negatively associated if for any disjoint subsets A, B of $\{1, \ldots, n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^{|A|}$ and g on $\mathbb{R}^{|B|}$,

$$Cov(f(X_k, k \in A),\ g(X_k, k \in B)) \le 0 \tag{1.1}$$

whenever the covariance exists, where $|A|$ denotes the cardinality of A.

This concept was introduced by Joag-Dev and Proschan (1983). It can be extended to $\mathbb{R}^d$-valued random vectors as follows (see Zhang, 2001). A finite sequence $\{X_1, \ldots, X_n\}$ of $\mathbb{R}^d$-valued random vectors is said to be negatively associated (or correlated) if for any disjoint subsets A, B of $\{1, \ldots, n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^{|A|d}$ and g on $\mathbb{R}^{|B|d}$,

$$Cov(f(X_k, k \in A),\ g(X_k, k \in B)) \le 0 \tag{1.2}$$

whenever the covariance exists. An infinite sequence of $\mathbb{R}^d$-valued random vectors is negatively associated if every finite subsequence is negatively associated.

Ko et al. in Ko et al. (2009) introduced the concept of negative association for random vectors with values in real separable Hilbert spaces. Let H be a real separable Hilbert space with orthonormal basis $\{e_j,\ j \in B\}$ and inner product $\langle \cdot, \cdot \rangle$. A sequence $\{X_n, n \ge 1\}$ of random vectors with values in H is said to be negatively associated if for any $d \ge 1$, the sequence of $\mathbb{R}^d$-valued random vectors $\{(\langle X_i, e_1 \rangle, \ldots, \langle X_i, e_d \rangle), i \ge 1\}$ is negatively associated.

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Limit theorems for dependent random vectors in Hilbert spaces and theirs applications are studied by a number of authors (see, e.g., Chen and White, 1998; Dehling et al., 2015; Politis and Romano, 1994 and the references therein). The dependent structures in these papers notably are mixing, near epoch dependence, and mixingales. A maximal inequality together with the Kolmogorov strong law of large numbers was established in Ko et al. (2009) for sequences of H-valued negatively associated random vectors. For a sequence of mean 0 negatively associated random vectors $\{X_n, n \geq 1\}$ in H, they proved that

$$E\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} X_i\right\|^2\right) \leq 2\sum_{i=1}^{n} E\|X_i\|^2, \quad n \geq 1. \tag{1.3}$$

(The constant 2 in the right hand side of (1.3) is missed in Ko et al., 2009, see the proof of Miao, 2012, Theorem 3.2.) The main result of Ko et al. (2009) was extended in Thanh (2013) by a different approach. In Thanh (2013), the conditions for the Marcinkiewicz–Zygmund type strong law were also provided.

In all papers mentioned above, the weak laws of large numbers were missed. In this paper, we provide conditions under which the random sums of negatively associated random vectors in Hilbert spaces obey the weak law of large numbers. Examples are provided to show that (a) the conditions cannot be dispensed within Theorem 2.1, and (b) under the conditions in Theorem 2.3, the corresponding $\mathcal{L}_2$ convergence does not prevail.

2. Weak laws of large numbers for random sums

Throughout this section, H will denote a real separable Hilbert space with orthonormal basis $\{e_j, j \in B\}$ and inner product $\langle\cdot,\cdot\rangle$. The symbol C denotes a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance. We assume $\{X_n, n \geq 1\}$ is a sequence of negatively associated random vectors in H. For $n \geq 1$, $k \geq 1$, $j \in B$, we set

$$X_k^{(j)} = \langle X_k, e_j\rangle,$$
$$Y_{nk}^{(j)} = -nI(X_k^{(j)} < -n) + X_k^{(j)} I(|X_k^{(j)}| \leq n) + nI(X_k^{(j)} > n),$$

and

$$Y_{nk} = \sum_{j\in B} Y_{nk}^{(j)} e_j.$$

For a sequence of random variables $\{T_n, n \geq 1\}$ and a sequence of positive constants $\{b_n, n \geq 1\}$, we write $T_n = \mathcal{O}_P(b_n)$ to indicate that T_n/b_n is *bounded in probability*; that is,

$$\lim_{K\to\infty}\sup_{n\geq 1} P\left\{\frac{|T_n|}{b_n} > K\right\} = 0.$$

It is easy to see that if $T_n/b_n \xrightarrow{P} a$ as $n \to \infty$ for some constant a, then $T_n = \mathcal{O}_P(b_n)$, but the reverse is not true.

In the following theorem, we establish the weak law of large numbers for non-random sums.

Theorem 2.1. *If*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j\in B} P(|X_k^{(j)}| > n) = 0 \tag{2.1}$$

and

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\sum_{j\in B} E((X_k^{(j)})^2 I(|X_k^{(j)}| \leq n))}{n^2} = 0, \tag{2.2}$$

then we obtain the weak law of large numbers

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - EY_{nk}) \xrightarrow{P} 0 \quad \text{as } n \to \infty. \tag{2.3}$$

Proof. Let $\varepsilon > 0$ be arbitrary. Then

$$P\left(\frac{1}{n}\left\|\sum_{k=1}^{n}(X_k - Y_{nk})\right\| > \varepsilon\right) \leq P\left(\bigcup_{k=1}^{n}(X_k \neq Y_{nk})\right)$$
$$\leq \sum_{k=1}^{n} P(X_k \neq Y_{nk})$$