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Quantizations of probability measures and preservation of the convex order

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ABSTRACT

Two probability measures admit a martingale transition if and only if they are ordered in the convex order (Kellerer, 1972). We show that the commonly used quantization method, L^2-quantization, does not have the property of preserving the convex order. We introduce an alternative quantization method and demonstrate that it preserves the convex order. This result has implications concerning the choice of quantization methods for the numerical construction of martingales with specified marginals.

1. Introduction

The construction of martingales having specified marginal laws is an interesting inverse problem (Madan and Yor, 2002; Baker and Yor, 2009; Albin, 2008; Hirsch et al., 2011). It is known that for this inverse problem to have a solution, it is necessary and sufficient that the specified marginal laws be ordered in the convex order (Kellerer, 1972). Existing construction methods work only in particular settings as they require the marginal laws to satisfy additional constraints. Through quantization of the marginal laws, the discretized version of this problem can be reduced to a linear programming problem, requiring no additional constraints on the marginal laws beyond the necessary ones for the existence of a martingale (Baker, 2012). However, as we will see in this paper the quantization method used must have the property of preserving the convex order. We will see that the commonly used L^2-quantization method does not have this property and will introduce one that does.

In the following we will be dealing with probability measures on $\mathbb{R}$.

Definition 1.1. A pair of probability measures (μ, ν) is said to admit a martingale transition if there exists a pair of random variables (X, Y) such that $X \sim \mu$, $Y \sim \nu$ and $\mathbb{E}[Y|X] = X$.

To offer the reader some perspective, we now briefly discuss the notion of martingale transitions in the broader context of constraining collections of measures. Imposing martingale transitions is a way of forcing measures. The idea of imposing martingale transitions took its origin in the works of Blackwell, Strassen and Kellerer (Blackwell, 1953; Strassen, 1965; Kellerer, 1972). Some kind or other of forcing structure is visible in several places in probability theory, not only for random variables but also for stochastic processes. For instance, much can be learned about Brownian motion by linking it to a Brownian bridge process, and working with suitable stopping times may also be seen as a kind of forcing on stochastic processes. More recently, Bruss and Yor (see Bruss and Yor, 2012) developed a new approach. They created what one could call a *martingale projection* which is close to the idea of a martingale transition for processes. They showed that an optimal

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stopping problem may make sense for an *unknown* stochastic process X by welding its past to a future *shadow process* which is determined by a martingale based on X. Coming back again to random variables, the most celebrated and probably best known way of forcing is the concept of coupling as a technique that allows one to compare two unrelated variables X and Y by welding them together (Lindvall, 2002).

Definition 1.2. Let μ and ν be probability measures on $\mathbb{R}$. The measure μ is said to be dominated by ν in the convex order (written $\mu \leq_{cx} \nu$), if for every convex function f the following inequality holds:

$$\int_{\mathbb{R}} f \, d\mu \leq \int_{\mathbb{R}} f \, d\nu.$$

Definition 1.3. A quantization of order n of a measure μ is a measure $\hat{\mu}$ which has a support consisting of at most n points.

When performing a quantization of a measure, the idea is (of course) that the measure $\hat{\mu}$ be a good approximation of the measure μ. The concept of martingale transitions is closely related to the concept of the convex order: in Kellerer (1972) it is shown that a pair of probability measures (μ, ν) admits a martingale transition if and only if $\mu \leq_{cx} \nu$. When searching for martingale transitions between two measures μ and ν, it can be useful to quantize both measures, and numerically construct martingale transitions between the quantized measures. For this to work, it is necessary and sufficient that the quantization method used have the property of preserving the convex order. Otherwise, a pair of probability measures which admits martingale transitions may no longer admit martingale transitions after it has been quantized.

The paper is organized as follows: In Section 1 we review several different characterizations of the convex order which we will use throughout the paper. In Section 2, we show that the commonly used L^2-quantization method does not preserve the convex order. In Section 3, we define an alternative quantization method, which we call $\mathcal{U}$-quantization, and prove that it has the property of preserving the convex order.

Definition 1.4. The potential function of a measure μ is given by

$$U\mu(t) = -\int_{\mathbb{R}} |t - x| d\mu(x).$$

The convex order can be established using potential functions or inverse cumulative distribution functions as shown by the following lemma, a proof of which can be found in Shaked et al. (1994).

Lemma 1.5. *Let μ and ν be measures with distribution functions F and G, the following are equivalent:*

(i) $\mu \leq_{cx} \nu$

(ii) $\int_{\mathbb{R}} |x - t| \, d\mu(x) \leq \int_{\mathbb{R}} |x - t| \, d\nu(x) \quad \forall \, t \in \mathbb{R}$

(iii) $\int_0^p F^{-1}(u) du \geq \int_0^p G^{-1}(u) du \quad \forall \, 0 \leq p \leq 1$

(iv) $U\mu(t) \geq U\nu(t)$

(v) *There exists random variables X and Y satisfying $X \sim \mu, \; Y \sim \nu$ and $\mathbb{E}[Y|X] = X$.*

To quantize a random variable X is to approximate it by a random variable $\hat{X}$ which has a support consisting of at most n points. The resulting quadratic error is given by:

$$\mathbb{E}|X - \hat{X}|^2.$$

The L^2 quantization of X is the random variable $\hat{X}$, supported on at most n points which minimizes the quadratic error.

Definition 1.6. Let μ be a probability measure on $\mathbb{R}$. Given a vector $(x_1, .., x_n) \in \mathbb{R}^n$, the *Voronoi quantization* of μ is defined as:

$$\hat{\mu} = \sum_{i=1}^{n} \mu(A_i) \, \delta_{x_i}$$

where δ_{x_i} denotes the Dirac point mass at x_i, and A_i is the Voronoi cell of x_i defined as:

$$A_i = \left\{ x \in \mathbb{R} : |x - x_i| \leq |x - x_j| \text{ for all } 1 \leq j \leq n \right\}.$$

(Voronoi quantization was originally developed for signal transmission purposes at the Bell Laboratory in the 1950s.)

The quadratic error of a Voronoi quantization is given by:

$$\sum_{i=1}^{n} \int_{A_i} |x_i - u|^2 \, d\mu(u).$$