Statistics and Probability Letters 100 (2015) 1–11

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Strong convergence of robust equivariant nonparametric functional regression estimators

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ARTICLE INFO

Article history:
Received 17 July 2014
Received in revised form 23 January 2015
Accepted 24 January 2015
Available online 3 February 2015

MSC:
62G35
62F12

Keywords:
Functional data
Kernel weights
M-location functionals
Robust estimation

ABSTRACT

Robust nonparametric equivariant M-estimators for the regression function have been extensively studied when the covariates are in $\mathbb{R}^k$. In this paper, we derive strong uniform convergence rates for kernel-based robust equivariant M-regression estimator when the covariates are functional.

1. Introduction

A common problem in statistics is to study the relationship between a random variable Y and a set of covariates X. In many applications, the covariates can be seen as functions recorded over a period of time and regarded as realizations of a stochastic process, often assumed to be in the L^2 space on a real interval. These variables are usually called functional variables in the literature. In this general framework, statistical models adapted to infinite-dimensional data have been recently studied. We refer to Ramsay and Silverman (2002, 2005), Ferraty and Vieu (2006) and Ferraty and Romain (2011) for a description of different procedures for functional data. In particular, linear nonparametric regression estimators in the functional setting, that is, estimators based on a weighted average of the response variables, have been considered, among others, by Ferraty and Vieu (2004) and Ferraty et al. (2006) who also considered estimators of the conditional quantiles. Burba et al. (2009) studied k-nearest neighbor regression estimators while Ferraty et al. (2010) obtained almost complete uniform convergence results (with rates) for kernel-type estimators.

However, in the functional case the literature on robust proposals for nonparametric regression estimation is sparse. Cadre (2001) studied estimation procedures for the L^1 median estimators for a random variable on a Banach space while Azzedine et al. (2008) studied nonparametric robust estimation methods based on the M-estimators introduced by Huber (1964), when the scale is known.

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http://dx.doi.org/10.1016/j.spl.2015.01.028
0167-7152/

In this paper, we consider the case in which the scale is unknown since in most practical situations, scale is unknown and needs to be estimated. As in location models, M-smoothers are shift equivariant. However, even if the mean and the median are scale equivariant, this property does not hold for M-location estimators unless a preliminary robust scale estimator is used to scale the residuals. The same holds for the robust nonparametric regression estimators considered in Azzedine et al. (2008). To ensure scale equivariance and robustness, a robust scale estimator needs to be used to decide which responses may be considered as atypical, so that their effect can be downweighted. In this sense, our contributions extend previous proposals in two directions. On one hand, we generalize the proposal given in the Euclidean case by Boente and Fraiman (1989) to provide robust equivariant estimators for the regression function in the functional case, that is, in the case where the covariates are in an infinite dimensional space. On the other hand, we extend the proposal given in Azzedine et al. (2008) to allow for an unknown scale and heteroscedastic models.

The paper is organized as follows. In Section 2, we state our notation, while in Section 3 we introduce the robust estimators to be considered. Section 4 contains the main results of this paper, that is, uniform convergence consistency and uniform convergence rates for the equivariant local M-estimators, over compact sets. These uniform convergence results are obtained either by giving conditions on the M-conditional location functional or by deriving similar results on the conditional empirical distribution function which extend those given in by Ferraty et al. (2010). Proofs are relegated to Appendix A and to the supplementary file available online (see Appendix B).

2. Basic definitions and notation

Throughout this paper, we consider independent and identically distributed observations (Y_i, X_i), $1 \leq i \leq n$ such that $Y_i \in \mathbb{R}$ and $X_i \in \mathcal{H}$ with the same distribution as (Y, X), where $(\mathcal{H}, d)$ is semi-metric functional space, that is, d satisfies the metric properties but $d(x, y) = 0$ does not imply $x = y$. We say that the observations satisfy a nonparametric functional regression model if (Y, X) is such that

$$Y = r(X) + U \tag{1}$$

where $r : \mathcal{H} \to \mathbb{R}$ is a smooth operator not necessarily linear. Throughout this paper, we will not require any moment conditions on the errors distribution. Usually in a robust framework, the error U is such that $U = \sigma(X)u$, where u is independent of X and with distribution F_0 symmetric around 0, that is, we assume that the errors u have scale equal to 1 to identify the function σ. When second moment exists, as it is the case of the classical approach, these conditions imply that $\mathbb{E}(U|X) = 0$ and $\textsc{Var}(U|X) = \sigma^2(X)$, which means that, in this situation, r and σ represent the conditional mean and standard deviation of the responses given the covariates, respectively. Hence, when $\mathbb{E}|Y| < \infty$, the regression function $r(X)$ in (1), which in this case equals $\mathbb{E}(Y|X)$, can be estimated using the extension to the functional setting of the Nadaraya–Watson estimator (see, for example, Härdle, 1990). To be more precise, let K be a kernel function and $h = h_n$ a sequence of strictly positive real numbers. Denote as

$$w_i(x) = K_i(x) \left(\sum_{i=1}^{n} K_i(x) \right)^{-1}, \tag{2}$$

where $K_i(x) = K\left(d(x, X_i)/h\right)$. Then, the classical regression estimator is defined as

$$\widehat{r}(x) = \sum_{i=1}^{n} w_i(x) Y_i. \tag{3}$$

Under regularity conditions, Ferraty and Vieu (2006) obtained convergence rates for the estimator $\widehat{r}(x)$, while Ferraty et al. (2010) derived uniform consistency results with rates for the estimator of the so-called generalized regression function $r_\varphi(x) = \mathbb{E}(\varphi(Y)|X = x)$ where φ is a known real Borel measurable function. As mentioned therein, this convergence is related to the Kolmogorov's ϵ-entropy of $S_{\mathcal{H}}$ and the function ϕ that controls the small ball probability of the functional variable X.

The conditional cumulative distribution function of Y given $X = x$ is defined, for each $x \in \mathcal{H}$, as $F(y|X = x) = \mathbb{P}(Y \leq y|X = x)$, for any $y \in \mathbb{R}$. As in Ferraty et al. (2010), we will assume that there is a regular version of the conditional distribution. An estimator of the conditional distribution function, can be obtained noting that $F(y|X = x) = \mathbb{E}(\mathbb{I}_{(-\infty,y]}(Y)|X = x)$, that is, taking $\varphi(Y) = \mathbb{I}_{(-\infty,y]}(Y)$ in the generalized regression function $r_\varphi(x)$ and using (3). Hence, the kernel estimator $\widehat{F}(y|X = x)$ of $F(y|X = x)$ equals

$$\widehat{F}(y|X = x) = \sum_{i=1}^{n} w_i(x) \mathbb{I}_{(-\infty,y]}(Y_i), \tag{4}$$

where $w_i(x)$ are defined in (2). Among other results, in Section 4, we obtain uniform strong convergence rates for $\widehat{F}(y|X = x)$ over $\mathbb{R} \times S_{\mathcal{H}}$ with $S_{\mathcal{H}} \subset \mathcal{H}$ a compact set, generalizing the results in Ferraty et al. (2010).

From now on, $m(x)$ stands for the median of the conditional distribution function, that is, $m(x) = \inf\left\{y \in \mathbb{R} : F(y|X = x) \geq \frac{1}{2}\right\}$. If $F(\cdot|X = x)$ is a strictly increasing distribution function, then the conditional median exists and is unique. Moreover,