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A strong limit theorem for the average of ternary functions of Markov chains in bi-infinite random environments

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ABSTRACT

We consider strong limit theorems for Markov chains in bi-infinite random environments. We first give a new proof of a strong limit theorem of Liu and Yang (1995) on the average of functions of non-homogeneous Markov chains by constructing a nonnegative martingale. As corollaries, for a Markov chain in a bi-infinite random environment, we obtain strong limit theorems for the conditional relative entropy and for the number of times that the Markov chain and related processes reach a given point.

1. Introduction

The limit theory for Markov chains is an important research topic of stochastic processes, in which many interesting results have been obtained (see, e.g., Meyn and Tweedie, 1993 and Revuz, 1984). As a natural generalization of classical time-homogeneous Markov chains, Cogburn (1980) introduced the concept of Markov chains in bi-infinite random environments, and published a series of papers (see Cogburn, 1980, 1984, 1990, 1991), in which the ergodic theory, the central limit theorem, the relationship between the direct convergence and the periodicity of transition function, and the existence of invariant measure for Markov chains in stationary environments are carefully studied in the framework of Hopf Markov chains. Orey (1991) proposed a series of interesting open problems. Seppalainen (1994) considered the large deviation theory of such Markov chains. Li (2001) studied their recurrence and transience, and proved the existence of invariant measure under the π-irreducible condition. In this paper we firstly give a strong limit theorem for the average of ternary functions of Markov chains in bi-infinite random environments. We then obtain some limit properties for these chains based on the strong limit theorem; in particular we obtain a limit theorem for their conditional relative entropy density.

An important question in information theory is the study of the limit properties of the relative entropy density. Shannon (1948), McMillan (1953) and Breiman (1957) proved, respectively, that, if a Markov chain is stationary and ergodic, then its relative entropy density converges in probability, L_1 and almost everywhere to a constant. Liu (1990) introduced an

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interesting analytic technique for the research of the relation between the relative entropy densities and a class of limit theorems for the sequence of m-valued random variables. The crucial part of the analytic technique is the application of Lebesgue's theorem on differentiability of monotone functions (see Liu, 1990). Subsequently, the author extends this analytic technique to the research of limit properties for non-homogeneous Markov chains (see Liu and Liu, 1994, Liu and Liu, 1996 and Liu and Yang, 1995). Although Liu's analytic technique can be applied to study the limit properties for non-homogeneous Markov chains, the process of proof using this analytic technique is complicated and difficult to be understood. In this paper, we introduce a new method to replace Liu's analytic technique. The core of this new method is the construction of a nonnegative martingale. The new method of proof we present in this paper is simpler and easier to be understood than the analytic technique proposed by Liu (1990).

The purpose of this paper is to study some limit properties of Markov chains in bi-infinite random environments. The main contribution is the introduction of a new method by the construction of a martingale, leading to a new proof of a strong law of large numbers of Liu and Yang (1995) on the average of functions for non-homogeneous Markov chains. We will state the theorem in the case of an arbitrary random environment to emphasize on our interest on the study of the random environment model, although the theorem holds for general non-homogeneous Markov chains. When the environment is stationary and ergodic, we give a simple moment condition under which the strong law of large numbers holds. As corollaries, for a Markov chain in a bi-infinite random environment, we obtain strong limit theorems for the conditional relative entropy and for the number of times that the Markov chain and related processes reach a given point.

The rest of the paper is organized as follows. The concept of Markov chains in bi-infinite random environments is presented in Section 2. The main results and the new method via the construction of a nonnegative martingale are presented in Section 3. The paper is concluded in Section 4.

2. Definitions and notations

Let $\mathcal{N}$ be the set of integers and $\mathcal{N}_+$ be the set of nonnegative integers, and $(\Omega, \Im, P)$ be the probability space. Let $\mathcal{X} = \{1, 2, \dots, N\}$, and $(\Theta, \mathcal{B})$ be an arbitrary measurable space. Let $\vec{\xi} := \{\xi_n, n = \dots, -1, 0, 1, \dots\}$ and $\vec{X} := \{X_n, n = 0, 1, 2, \dots\}$ be random sequences defined on $(\Omega, \Im, P)$ with value in Θ and $\mathcal{X}$, respectively. Let $\{P(\theta), \theta \in \Theta\}$ be a family of transition functions, that is, $\{P(\theta)\}$ be a family of transition function for a given $\theta \in \Theta$. For any sequence $\vec{\eta} = \{\eta_n\}$, write $\vec{\eta}_k^r := \{\eta_n, k \le n \le r\}, -\infty \le k \le r \le \infty$. Write $\mathcal{F}_n := \sigma\{X_0, \dots, X_n, \vec{\xi}\}, n \ge 0$, and $\vec{\Theta} = \Pi_{j=-\infty}^{\infty}\Theta_j$, where $\Theta_j = \Theta, j = \dots, -1, 0, 1, \dots$. Let T be the shift operator such that for any $\vec{\xi}, T^k\vec{\xi} = \{\eta_n\}$ with $\eta_n = \xi_{n+k}, n = \dots, -1, 0, 1, \dots$.

Definition 2.1. If for any $x, y \in \mathcal{X}$ and $n \in \mathcal{N}_+$,

$$P(X_0 = x|\vec{\xi}) = P(X_0 = x|\vec{\xi}_{-\infty}^{0}),$$
$$P(X_{n+1} = y|\mathcal{F}_n) = P(\xi_n; X_n, y),$$

then we call $\vec{X}$ a Markov chain in the bi-infinite random environment $\vec{\xi}$.

From here on $\vec{X}$ is a Markov chain in the bi-infinite random environment $\vec{\xi}$, its n-step transition matrix is

$$P_n = \{P(\theta_0, \dots, \theta_{n-1}; x, y),\ x, y \in \mathcal{X}, \theta_0, \dots, \theta_{n-1} \in \Theta\}, \tag{2.1}$$

where $P(\theta_0, \dots, \theta_{n-1}; x, y) = \prod_{m=0}^{n-1} P(\theta_m)(x, y)$.

By definition, $\{(X_n, T^n\vec{\xi}), n \ge 0\}$ is a Markov bi-chain (see Cogburn, 1984, 1990, 1991 and Li, 2001). Let

$$P(\vec{\theta}; x_0, \dots, x_n) \equiv \prod_{m=1}^{n} P(\theta_{m-1}; x_{m-1}, x_m), \tag{2.2}$$

it states that, given $\vec{\xi} = \vec{\theta}$, the transition probability of the sample paths of the chain $\{X_n, n \ge 0\}$ evolve from x_0 to x_1, to $x_2, \dots$, and to x_n.

Let

$$g_n(\omega) = -\frac{1}{n}\log P(\vec{\xi}; X_0, \dots, X_n)$$

$\{g_n(\omega), n \ge 0\}$ is called the conditional relative entropy density of $\{X_n, n \ge 0\}$ given ω. By (2.2), we have

$$g_n(\omega) = -\frac{1}{n}\sum_{m=1}^{n}\log P(\xi_{m-1}; X_{m-1}, X_m). \tag{2.3}$$

3. Limit theorems via the martingale approach

In this section, we present our new method via the construction of a nonnegative martingale, to prove a limit theorem for the average of ternary functions of Markov chains in bi-infinite random environments. As corollaries, for a Markov chain in a bi-infinite random environment, we obtain strong limit theorems for the conditional relative entropy and for the number of times that the Markov chain and related processes reach a given point.