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Expansions for bivariate copulas

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ABSTRACT

About twenty expansions applicable for a wide range of bivariate copulas are given. These expansions being mostly elementary enable easy computation of measures and properties of copulas.

1. Introduction

The concept of copulas was introduced by Sklar (1959). Since then many parametric, non-parametric and semi-parametric models have been proposed for copulas, including methods for constructing models for copulas. Most of the proposed models have been parametric models.

Applications of copulas are too numerous to list. Some recent applications have been simulation of multivariate sea storms (Corbella and Stretch, 2013), dependence structure between the stock and foreign exchange markets (Wang et al., 2013) and operational risk management (Arbenz, 2013). Most applications have been based on parametric models for copulas.

However, there has not been convenient tools to compute parametric copulas or measures associated with them. The aim of this note is to present a convenient tool applicable for a wide class of known copulas.

Let $C\left(u_1, u_2, \ldots, u_d\right)$ denote a copula. Let $c\left(u_1, u_2, \ldots, u_d\right)$ denote its joint density, provided it exists. We suppose $C\left(u_1, u_2, \ldots, u_d\right)$ can be expanded as

$$C\left(u_1, u_2, \ldots, u_d\right) = \sum_{i=1}^{n} \alpha_i u_1^{a_{1,i}} u_2^{a_{2,i}} \cdots u_d^{a_{d,i}}, \tag{1}$$

where $n \geq 1$ is an integer and $\left\{\left(\alpha_i, a_{1,i}, a_{2,i}, \ldots, a_{d,i}\right) : i \geq 1\right\}$ are real numbers. So,

$$c\left(u_1, u_2, \ldots, u_d\right) = \sum_{i=1}^{n} \alpha_i a_{1,i} a_{2,i} \cdots a_{d,i} u_1^{a_{1,i}-1} u_2^{a_{2,i}-1} \cdots u_d^{a_{d,i}-1}. \tag{2}$$

In spite of their simplicity, (1) and (2) have not been noted in the literature. We will see that (1) and (2) allow for easy computation of most measures and properties (but not all) associated with copulas.

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In Section 2, we shall show (1) and (2) hold for a wide class of known copulas. In Section 3, we derive expansions for many measures and properties associated with bivariate versions of (1) and (2):

$$C(u,v)=\sum_{i=1}^{n}\alpha_i u^{a_i} v^{b_i} \tag{3}$$

and

$$c(u,v)=\sum_{i=1}^{n}\alpha_i a_i b_i u^{a_i-1} v^{b_i-1} \tag{4}$$

for $n \geq 1$ an integer and $\{(\alpha_i, a_i, b_i) : i \geq 1\}$ some real numbers. Most of these expansions are elementary, so allowing for widespread application. An illustration of the expansions is provided in Section 4. The proofs for the expansions in Section 3 are outlined in Section 5.

The readers should not confuse the aim of this note with finding conditions on $\left\{\left(\alpha_i, a_{1,i}, a_{2,i}, \ldots, a_{d,i}\right) : i \geq 1\right\}$ such that (1) is a valid copula. The aim of this note is to show that a wide class of known copulas can be expressed in the form (1) and hence the derived measures and properties apply to that wide class.

2. Examples

There are many, many copulas which can be expressed in the form (1). Here, we list just five of them. For some others, we refer the readers to Nelsen (2006).

Example 1. The Farlie–Gumbel–Morgenstern copula (Nelsen, 2006) defined by

$$C\left(u_1,\ldots,u_d\right)=u_1\cdots u_d\left[1+\sum_{k=2}^{d}\sum_{1\leq j_1<\cdots<j_k\leq d}\theta_{j_1,\ldots,j_k}\left(1-u_{j_1}\right)\cdots\left(1-u_{j_k}\right)\right]$$

satisfies (1) for $-1 \leq \theta_{j_1,\ldots,j_k} \leq 1$ for all $j_1, \ldots, j_k$.

Example 2. Ibragimov (2009)'s copula defined by

$$C\left(u_1,\ldots,u_d\right)=\prod_{i=1}^{d}u_i\left[1+\sum_{c=2}^{d}\sum_{1\leq i_2<\cdots<i_c\leq d}a_{i_1,\ldots,i_c}\left(u_{i_1}^{\ell}-u_{i_1}^{\ell+1}\right)\cdots\left(u_{i_c}^{\ell}-u_{i_c}^{\ell+1}\right)\right]$$

satisfies (1) for $-\infty < a_{i_1,\ldots,i_c} < \infty$ such that

$$\sum_{c=2}^{d}\sum_{1\leq i_2<\cdots<i_c\leq d}\left|a_{i_1,\ldots,i_c}\right| \leq 1.$$

Example 3. The cubic copula (Nelsen, 2006) defined by

$$C(u,v)=uv\left[1+\alpha(u-1)(v-1)(2u-1)(2v-1)\right]$$

satisfies (1) for $-1 \leq \alpha \leq 2$.

Example 4. Let $u_{(1)} \leq u_{(2)} \leq \cdots \leq u_{(d)}$ denote sorted values of $u_1, u_2, \ldots, u_d$ and let a_i, $i = 0, 1, \ldots, d-1$ denote some real numbers. Mai and Scherer (2009)'s copula defined by

$$C\left(u_1,\ldots,u_d\right)=\prod_{i=1}^{d}u_{(i)}^{a_{i-1}}$$

satisfies (1) for $a_0 = 1$ and $\{a_i\}$ d-monotone; that is, $\Delta^{j-1}a_k = 0$ for all $k = 0, 1, \ldots, d-1$ and $j = 1, 2, \ldots, d-k$, where

$$\Delta^j a_k=\sum_{i=0}^{j}(-1)^i\binom{j}{i}a_{k+i}$$

for $j \geq 0$ and $k \geq 0$.