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On kernel estimators of density for reversible Markov chains

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ABSTRACT

In this paper we investigate the kernel estimator of the density for a stationary reversible Markov chain. The proofs are based on a new central limit theorem for a triangular array of reversible Markov chains obtained under conditions imposed to covariances, which has interest in itself.

1. Introduction and main results

For estimating the marginal density for dependent sequences the dependence structure plays an important role. One possible estimator is the kernel estimator introduced by Rosenblatt (1956a). In general the dependence is imposed in terms of mixing conditions (Bradley, 1993; Bosq et al., 1999, among many others), in terms of coupling coefficients for functions of i.i.d. (Wu et al., 2010) or positive association of random variables (Lin, 2003).

In this paper we study the kernel estimator for reversible Markov chains. It is well known that for strictly stationary reversible Markov chains the covariances can be viewed as a measure of dependence (see Kipnis and Varadhan, 1986). When estimating the density via kernel estimators we introduce a triangular array of random variables which is only row-wise stationary. This makes it difficult for studying the kernel density of the marginal distribution for reversible Markov chains without imposing recurrence conditions. As a matter of fact results on the kernel estimators for marginal density of reversible Markov chains are very rare. We noticed only the paper by Lei (2006) dealing with large deviations results for the integrated error of the kernel density estimators for reversible Markov chains. The class they considered is of reversible irreducible Markov chains with the transitions satisfying a uniform integrability condition in square mean. However their result cannot be applied when studying the density at a point or several points. In this paper we develop tools that make this study possible.

Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary reversible Markov chain with marginal distribution $\pi(A) = P(X_n \in A)$, for all Borel sets A. For a stationary Markov chain the reversibility means that the distribution of (X_0, X_1) is the same as of (X_1, X_0). Assume that

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π has a marginal density $f(x)$, continuous at x. We shall consider in this paper the Rosenblatt (1956a) estimator of density defined by

$$\hat{f}_n(x) = \frac{1}{nb_n} \sum_{k=1}^{n} K\left(\frac{x - X_k}{b_n}\right), \tag{1}$$

where b_n is a bandwidth converging to 0 and K is a kernel, a known density function.

The problem considered in this paper is the consistency and the speed of convergence of the kernel density estimator of the density at several points $(x_j)_{1\le j\le m}$ which will be given via a multivariate CLT.

To treat the problem we shall use the following notation

$$H_k(u, v) = P(X_0 > u, X_k > v) - P(X_0 > u)P(X_k > v) \tag{2}$$

and we shall denote

$$\eta_k = \iint |H_k(u, v)|dudv. \tag{3}$$

The condition we shall impose to η_k is

$$\eta_k \le \frac{1}{k^4 l(k)}, \tag{4}$$

where $l(x)$ is a function increasing to infinity such that for any positive k, $\lim_{x\to\infty} l(kx)/l(x) = 1$ (slowly varying at infinity).

The following condition is imposed to the joint density of the vector (X_0, X_2) and a family of points of interest $(x_j)_{1\le j\le m}$: there exists the joint density $f_2(x, y)$ of (X_0, X_2) which is locally bounded around any pair $(x_i, x_j)_{1\le i,j\le m}$ in the sense that there exists a constant M and a constant C_M (both depending on (x_i, x_j)) such that

$$\sup_{|a|<M} |f_2(x_i + a, x_j + a)| < C_M. \tag{5}$$

This condition is weaker than the condition which is usually imposed in the dependent cases which requires that all the densities of vectors (X_0, X_j) are uniformly bounded on $\mathbb{R}^2$ (see condition in Bosq, 1998, or in Bosq et al., 1999). Local conditions can be found for instance in papers by Liebscher (1999) and Dedecker and Merlevède (2002).

All along the paper, we assume that the kernel K satisfies the Condition C below:

(C1) K is symmetric decreasing on $(0, \infty)$ and $\int K(u)du = 1$.
(C2) $x^2 K(x) \to 0$ as $x \to \infty$.
(C3) K is differentiable with $K'(x)$ bounded.

Note that a normal kernel will satisfy all these conditions. The convergence in distribution will be denoted by $\Rightarrow$, and $\xrightarrow{P}$ denotes the convergence in probability.

The main result of this paper is the following:

Theorem 1. *Let $(X_j)_{j\in\mathbb{Z}}$ be a stationary reversible Markov chain with marginal density function $f(x)$ satisfying condition (4). Assume that the bandwidth b_n in the estimator (1) satisfies $nb_n^4 \to \infty$ and the kernel K satisfies Condition C. Then, at any points $x_1, \dots, x_m$ where $f(x)$ is continuous, different of 0 and the joint densities satisfy condition (5), we have*

$$\sqrt{nb_n}\left(\frac{\hat{f}_n(x_j) - \mathbb{E}\hat{f}_n(x_j)}{(\hat{f}_n(x_j)\int K^2(u)du)^{1/2}}, 1 \le j \le m\right) \Rightarrow N(0, I_m),$$

where I_m is the identity matrix.

It is well known that if the density is twice continuously differentiable at x_j then the bias is of order (see Härdle, 1991, relation (2.3.2))

$$\mathbb{E}(\hat{f}_n(x_j)) - f(x_j) = \frac{b_n^2}{2} f''(x_j) + o(b_n^2) \quad \text{as } b_n \to 0.$$

By combining this result with Theorem 1 we get the following corollary:

Corollary 2. *In addition to the conditions of Theorem 1, assume that f is twice continuously differentiable at $(x_j)_{1\le j\le m}$ and $nb_n^5 \to 0$. Then*

$$\sqrt{nb_n}\left(\frac{\hat{f}_n(x_j) - f(x_j)}{(\hat{f}_n(x_j)\int K^2(u)du)^{1/2}}, 1 \le j \le m\right) \Rightarrow N(0, I_m).$$

Let us comment about the dependence coefficient used in our results defined in (3).