Statistics and Probability Letters 100 (2015) 158–163

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

On the distribution of a max-stable process conditional on max-linear functionals

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ARTICLE INFO

Article history:
Received 18 September 2014
Received in revised form 27 January 2015
Accepted 2 February 2015
Available online 19 February 2015

Keywords:
Conditional simulation
Normalized representation
Spatial extremes

ABSTRACT

Recently, Dombry and Éyi-Minko (2013) provided formulae for the distribution of a max-stable process conditional on its values at given sites and proposed a methodology for sampling from this distribution. We generalize their results by allowing for conditions stemming from max-linear functionals of the process. Furthermore, we show that the conditional distribution of the extremal functions, i.e. the spectral functions attaining the imposed conditions, is closely related to the normalized spectral representation. The results are illustrated in several examples.

1. Introduction

During the last years, max-stable processes have become frequently used models for spatial extremes, in particular for applications in environmental sciences. In the context of the prediction of these processes given some data, the question of their conditional distribution arises. The conditions considered so far are restricted to values of the process at several sites. For this case, exact formulae in terms of the exponent measure of the max-stable process have been provided (Dombry and Éyi-Minko, 2013) and explicit computations have been implemented for several subclasses (cf. Dombry et al., 2013; Oesting and Schlather, 2014, for example).

In this paper, we analyze the conditional distribution allowing for more general conditions given by max-linear functionals of the process. For instance, a condition on the maximum of the process may be considered. In this case, the analysis of the conditional distribution may provide further insight in characteristics of extreme events that exceed a certain value. More precisely, we consider a max-stable process $\{Z(x),\ x \in K\}$ on some compact set $K \subset \mathbb{R}^d$ which – without loss of generality – can be assumed to have unit Fréchet marginals, i.e. $\mathbb{P}(Z(x) \le z) = \exp(-1/z)$, $z > 0$, for all $x \in K$. Further, we require Z to be sample-continuous, that is, all sample paths are in the space $C_+(K)$ of nonnegative continuous functions on K. Thus, Z possesses a spectral representation (see de Haan, 1984; Giné et al., 1990; Penrose, 1992, for example):

$$Z(t) = \max_{i \in \mathbb{N}} U_i W_i(t), \quad t \in K, \tag{1}$$

where $\{U_i,\ i \in \mathbb{N}\}$, is a Poisson point process on $(0, \infty)$ with intensity measure $u^{-2}\,\mathrm{d}u$ and W_i, $i \in \mathbb{N}$, are independent copies of a nonnegative sample-continuous stochastic process W with $\mathbb{E}W(t) = 1$ for all $t \in K$.

Assume that we observe values of continuous max-linear functionals $L_1, \ldots, L_n : C_+(K) \to [0, \infty)$, i.e.

$$L_j(\max\{a_1 f_1, a_2 f_2\}) = \max\{a_1 L_j(f_1), a_2 L_j(f_2)\}, \quad \text{for all } a_1, a_2 \ge 0,\ f_1, f_2 \in C_+(K),\ j = 1, \ldots, n.$$

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http://dx.doi.org/10.1016/j.spl.2015.02.002
0167-7152/

An example for such max-linear functionals are $L(f) = \sup_{t\in K'} f(t)$ for some compact set $K' \subset K$, including the special cases $L(f) = \sup_{t\in K} f(t)$ and $L(f) = f(t_0)$ for some $t_0 \in K$. More generally, using continuity arguments and the compactness of K, it can be shown that every functional of this type is of the form

$$L(f) = \sup_{t\in K} h(t)f(t), \quad f \in C_+(K), \tag{2}$$

for some bounded, but not necessarily continuous function $h : K \to [0, \infty)$.

In this paper, we analyze the conditional distribution of $Z \mid \mathbf{L}(Z)$ where $\mathbf{L}(Z) = (L_1(Z), \ldots, L_n(Z))^\top$. In Section 2, we provide formulae for the conditional distribution in terms of the exponent measure generalizing the results of Dombry and Éyi-Minko (2013). More explicit expressions for the case of one single condition are derived in Section 3 making use of connections to the normalized spectral representations to max-stable processes. Section 4 deals with the more general case of a finite number of conditions.

2. General theory

In the following, we analyze the distribution of Z conditionally on $\mathbf{L}(Z) = \mathbf{z}$ for some $\mathbf{z} = (z_1, \ldots, z_n)^\top \in (0, \infty)^n$. Here, we note that, because of the max-linearity of $L_j, j = 1, \ldots, n$, the properties of $L_j(Z)$ are directly connected to those of $L_j(W)$. By Prop. 2.3 in Oesting et al. (2013), the finiteness of $L(Z)$ implies that $\mathbb{E}L_j(W) < \infty$ and $\mathbb{P}(L_j(Z) \le z) = \exp(-\mathbb{E}(L_j(W))/z), z > 0$, i.e., $L_j(Z)$ follows a Fréchet distribution. To exclude the trivial case that the distribution of $L_j(Z)$ is degenerate, assume that $p_j = \mathbb{P}(L_j(W) > 0) > 0$ for every $j \in \{1, \ldots, n\}$. We consider the extended process $Z_L = (\{Z(t), t \in K\}, \mathbf{L}(Z))$ on $C_+(K) \times (0, \infty)^n$. By the max-linearity of L_j, we obtain that $L_j(Z) = \max_{i\in\mathbb{N}} U_i L_j(W_i)$. Thus,

$$Z_L = \max_{i\in\mathbb{N}} (\xi_i, \mathbf{L}(\xi_i)) \tag{3}$$

where the maximum is considered componentwise and $\Pi = \sum_{i\in\mathbb{N}} \delta_{(\xi_i, \mathbf{L}(\xi_i))}$ denotes a Poisson point process on $S = C_+(K) \times [0, \infty)^n$ with intensity measure

$$\Lambda(A \times B) = \int_0^\infty u^{-2}\mathbb{P}(uW \in A, u\mathbf{L}(W) \in B)\, \mathrm{d}u,$$

for Borel sets $A \subset C_+(K)$ and $B \subset [0, \infty)^n$ (cf. Kingman, 1993), that is, ξ_i corresponds to the product $U_i W_i$ in representation (1).

Perceiving the conditions $L_1(Z) = z_1, \ldots, L_n(Z) = z_n$ as conditions on the value of the process Z_L at specific “sites” according to representation (3) the results of Dombry and Éyi-Minko (2013) can be applied on Z_L, in order to derive the distribution of Z conditional on $\mathbf{L}(Z)$. To this end, for every non-empty index subset $J \subset \{1, \ldots, n\}$, we consider the J-extremal random point measure Π_J^+ and the J-subextremal random point process Π_J^-, defined by

$$\Pi_J^+ = \sum_{i\in\mathbb{N}} \delta_{\xi_i} \mathbf{1}_{\{L_j(\xi_i)=L_j(Z) \text{ for some } j\in J\}} \quad \text{and} \quad \Pi_J^- = \sum_{i\in\mathbb{N}} \delta_{\xi_i} \mathbf{1}_{\{L_j(\xi_i)<L_j(Z) \text{ for all } j\in J\}}.$$

It can be shown that Π_J^+ and Π_J^- are well-defined point processes on $C_+(K)$ (see Dombry and Éyi-Minko, 2013, Lemma A.3). Further, as $\Pi_{\{j\}}^+(C_K^+) = 1$ a.s. (cf. Dombry and Éyi-Minko, 2013, Prop. 2.5), $\Pi_{\{1,\ldots,n\}}^+$ is characterized via so-called hitting scenarios (cf. Wang and Stoev, 2011; Dombry and Éyi-Minko, 2013), i.e. partitions $\tau = \{\tau_1, \ldots, \tau_l\}$ of $\{1, \ldots, n\}$ representing the situation that $\Pi_{\{1,\ldots,n\}}^+ = \{\xi_1^+, \ldots, \xi_l^+\}$ s.t.

$$L_j(\xi_k^+) \begin{cases} = L_j(Z), & j \in \tau_k \\ < L_j(Z), & j \notin \tau_k \end{cases}, \quad 1 \le j \le n, \ 1 \le k \le l.$$

Let $\Theta \in \mathcal{P}_{\{1,\ldots,n\}}$ be the random partition realized by Z where $\mathcal{P}_{\{1,\ldots,n\}}$ denotes the space of partitions of $\{1, \ldots, n\}$. Based on the different hitting scenarios, conditional simulations of $Z \mid \mathbf{L}(Z) = \mathbf{z}$ can be performed via the following three-step procedure proposed by Dombry and Éyi-Minko (2013):

1. Draw a partition $\tau = \{\tau_k\}_{k=1}^l$ from the distribution of $\Theta \mid \mathbf{L}(Z) = \mathbf{z}$.
2. Simulate $(\xi_k^+)_{k=1}^l$ as a realization of $\Pi_{\{1,\ldots,n\}}^+ \mid \mathbf{L}(Z) = \mathbf{z}, \ \Theta = \tau$.
3. Draw $\{\xi_i^-\}_{i\in\mathbb{N}}$ from the distribution of $\Pi_{\{1,\ldots,n\}}^- \mid \mathbf{L}(Z) = \mathbf{z}$.

Then, $\max_{k=1}^l \xi_k^+ \vee \max_{i\in\mathbb{N}} \xi_i^-$ is a realization of $Z \mid \mathbf{L}(Z) = \mathbf{z}$.

The distributions involved in the algorithm are given in the following theorem which summarizes Theorems 3.1 and 3.2 in Dombry and Éyi-Minko (2013). First, we need some more notation. For a non-empty index subset $J \subset \{1, \ldots, n\}$, let $R_J : [0, \infty)^n \to [0, \infty)^{|J|}$ be the projection on the components belonging to J, i.e. $R_J(\mathbf{z}) = \mathbf{z}_J$ where $\mathbf{z}_J = (z_j)_{j\in J}$ for $\mathbf{z} = (z_j)_{j=1}^n$. Further, let μ be the exponent measure of Z, i.e. $\mu(A) = \Lambda(A \times [0, \infty)^n), A \subset C_+(K)$, and μ_J the exponent