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Elicitable distortion risk measures: A concise proof

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ABSTRACT

Elicitability has recently been discussed as a desirable property for risk measures. Kou and Peng (2014) showed that an elicitable distortion risk measure is either a Value-at-Risk or the mean. We give a concise alternative proof of this result, and discuss the conflict between comonotonic additivity and elicitability.

1. Distortion risk measures

We consider a standard atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by D the set of distribution functions on $\mathbb{R}$. A law invariant risk measure ρ is a mapping from D_ρ to $[-\infty, +\infty]$, where $D_\rho \subset D$. Write

$$H = \{h : [0, 1] \to [0, 1] : h \text{ is non-decreasing}, h(0) = 0 \text{ and } h(1) = 1\}.$$

Definition 1.1. A *distortion risk measure* $\rho : D_\rho \to \mathbb{R}$ is defined by

$$\rho(F) = \int_{-\infty}^{0} (h(1 - F(x)) - 1)\mathrm{d}x + \int_{0}^{\infty} h(1 - F(x))\mathrm{d}x, \tag{1.1}$$

where D_ρ is a set of some $F \in D$ such that (1.1) is well-defined, and $h \in H$ is called the *distortion function* of ρ.

The two most popular risk measures used in practice, *Value-at-Risk* (VaR) and *Expected Shortfall* (ES), are both distortion risk measures; for a recent discussion on VaR and ES, see Embrechts et al. (2014). We refer to Wang et al. (1997), Acerbi (2002), Kusuoka (2001) and Kou and Peng (2014) for more details and examples of distortion risk measures. Distortion risk measures are closely related to L-statistics (linear combinations of rank statistics, introduced for robust estimation); see Chapter 3 of Huber and Ronchetti (2009).

2. Elicitability

Elicitability was originally introduced as a property of set-valued functions. For consistency, we consider $\rho : D_\rho \to 2^{\mathbb{R}}$ as set-valued functions. This includes, for example, the case of quantiles, which may be an interval. In most cases, each value of ρ is a set with exactly one element as in Section 1, and we simply treat them as mappings to $\mathbb{R}$.

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Definition 2.1. A scoring function $S : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is called *consistent* for ρ *with respect to* D_ρ, if

$$\mathbb{E}(S(t, X)) \leqslant \mathbb{E}(S(x, X)) \tag{2.1}$$

for all $t \in \rho(F)$ and all $x \in \mathbb{R}$, where X is a random variable with distribution $F \in D_\rho$. We speak of *strict consistency* of S if equality in (2.1) implies $x \in \rho(X)$. The functional ρ is *elicitable* (*with respect to* D_ρ) if there exists a strictly consistent scoring function for it.

Roughly speaking, the forecasting of an elicitable risk measure can be evaluated using a score function, whereas there is no clear criterion to evaluate the forecasting of a non-elicitable risk measure. Arguments for the desirability of elicitability for risk management and other statistical sciences can be found in Gneiting (2011) and Ziegel (in press).

3. Elicitable distortion risk measures

A necessary condition for a functional to be elicitable is convex level sets, that is, if $t \in \rho(F) \cap \rho(G)$, then

$$t \in \rho(\lambda F + (1-\lambda)G),$$

whenever $\lambda F + (1-\lambda)G \in D_\rho$ for $\lambda \in [0, 1]$; see Osband (1985). Developments on risk measures with convex level sets can be found in Weber (2006), Lambert (2012), Ziegel (in press), Bellini and Bignozzi (in press), Kou and Peng (2014) and Delbaen et al. (2014). The work of Steinwart et al. (2014) shows that convex level sets are also a sufficient criterion for elicitability under some weak regularity assumptions on ρ; see also Lambert (2012).

The work of Weber (2006) investigated monetary risk measures with convex level sets under some additional regularity assumptions in a context of dynamic consistency. Under his assumptions, convex level sets are necessary and sufficient for the risk measure to be a shortfall risk measure. In the case of convex risk measures, Delbaen et al. (2014) show that Weber's (2006) assumptions are equivalent to the weak compactness property. They extend his result by showing that all convex risk measures with convex level sets are necessarily *generalized* shortfall risk measures. Bellini and Bignozzi (in press) considered monetary elicitable risk measures based on the results of Weber (2006). They use a more restrictive definition of elicitability by imposing regularity conditions on the scoring function S, which in turn ensures that Weber's assumptions are satisfied. It was shown that (1) a monetary risk measure is elicitable only if it is a shortfall risk measure; (2) a convex risk measure is elicitable if and only if it is a convex shortfall; (3) a coherent risk measure is elicitable if and only if it is an expectile. In the case of coherent risk measures it is possible to show directly from the Kusuoka representation (Kusuoka, 2001), that the only coherent risk measures with convex level sets are expectiles; see Ziegel (in press). While it is possible to apply Weber's (2006) results to distortion risk measures, this requires unnecessary additional assumptions, which can be avoided by exploiting the structure of distortion risk measures directly.

The following result (Kou and Peng, 2014, Theorem A.1) characterizes distortion risk measures with convex level sets, which leads to a full characterization of elicitable distortion risk measures. We provide an alternative proof of this result, which is substantially shorter and less technical.

Theorem 3.1 (*Kou and Peng, 2014*)**.** *Let D^* be the class of distributions with finite support and ρ be a distortion risk measure with distortion function $h \in H$ as defined at (1.1) whose restriction to D^* has convex level sets. Then h is either the identity on* $[0, 1]$ *or it is of the form*

$$h(x) = \begin{cases} 0, & x \in [0, \alpha), \\ c, & x = \alpha, \\ 1, & x \in (\alpha, 1], \end{cases} \tag{3.1}$$

for some $\alpha, c \in [0, 1]$. If $\alpha = 0$ or $\alpha = 1$, then $c = 0$ or $c = 1$, respectively.

Proof. Let $0 < x < y$, $\lambda \in [0, 1]$. Then

$$\rho((1-\lambda)\delta_x + \lambda\delta_y) = x + h(\lambda)(y - x),$$

where δ_x is the Dirac measure at the point $x \in \mathbb{R}$. In particular, $\rho(\delta_1) = 1$. Let $\lambda \in [0, 1]$ such that $h(\lambda) > 0$. All $0 < x < y$ such that $\rho((1-\lambda)\delta_x + \lambda\delta_y)$ are characterized by the equation

$$y = \left(1 - \frac{1}{h(\lambda)}\right)x + \frac{1}{h(\lambda)}.$$

In order to obtain $x < y$, we need to choose $x < 1$, which then implies $y > 1$.

Convexity of level sets on D^* now implies for all $p \in [0, 1]$, $0 < x < 1 < y$ chosen as described before, that

$$\begin{aligned} 1 &= \rho\big(p((1-\lambda)\delta_x + \lambda\delta_y) + (1-p)\delta_1\big) \\ &= x + h(1 - p(1-\lambda))(1-x) + h(\lambda p)\left(1 - \frac{1}{h(\lambda)}\right)(1-x), \end{aligned}$$