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Partial stochastic dominance for the multivariate Gaussian distribution

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ABSTRACT

We establish a partial stochastic dominance result for the maximum of a multivariate Gaussian random vector with positive intraclass correlation coefficient and negative expectation. Specifically, we show that the distribution function intersects that of a standard Gaussian exactly once.

1. Introduction

Gaussian comparison inequalities provide a useful tool in probability and statistics, with applications in areas including Gaussian processes and extreme value theory. A survey of results and applications can be found in the books by Ledoux and Talagrand (1991) and Lifshits (1995). Suppose that $\boldsymbol{X} = (X_1, \dots, X_k)$ and $\boldsymbol{Y} = (Y_1, \dots, Y_k)$ are two multivariate Gaussian vectors. Comparison inequalities typically involve finding conditions on the correlation structures of $\boldsymbol{X}$ and $\boldsymbol{Y}$ from which it can be deduced that $\mathbb{P}(\boldsymbol{X} \in C) \le \mathbb{P}(\boldsymbol{Y} \in C)$ for some suitable class of sets $C \in \mathbb{R}^k$, usually of the form $\prod_{i=1}^{k}(-\infty, x_i]$. An important example is Slepian's inequality (Slepian, 1962) which states that if $\mathbb{E}(\boldsymbol{X}) = \mathbb{E}(\boldsymbol{Y})$, $\mathbb{E}(X_i^2) = \mathbb{E}(Y_i^2)$ for all i and $\mathbb{E}(X_iX_j) \le \mathbb{E}(Y_iY_j)$ for all $i \ne j$, then $\mathbb{P}(X_1 \le x_1, \dots, X_k \le x_k) \le \mathbb{P}(Y_1 \le x_1, \dots, Y_k \le x_k)$ for all $(x_1, \dots, x_k) \in \mathbb{R}^k$.

A direct consequence of Slepian's inequality is that $F_{X^*}(x) \le F_{Y^*}(x)$ for all $x \in \mathbb{R}$, where $X^* = \max\{X_1, \dots, X_k\}$ and $Y^* = \max\{Y_1, \dots, Y_k\}$, so X^* stochastically dominates Y^* and the distribution functions of X^* and Y^* never cross each other. In this paper, by contrast, we obtain a *partial* stochastic dominance result by showing that, under certain assumptions on $\boldsymbol{X}$, $F_{X^*}(x)$ intersects the standard Gaussian distribution function $\Phi(x)$ *exactly once*. Suppose that $\boldsymbol{X}$ is a multivariate normal random vector with expectation $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$ and all variances equal to 1. It is easy to see that if $\mu_i \ge 0$ for some i, then $F_{X^*}(x) < \Phi(x - \mu_i) \le \Phi(x)$. Therefore, X^* dominates the standard Gaussian. Our result shows that when $\mu_i < 0$ for all i, if the covariances of $\boldsymbol{X}$ are equal and positive (that is, $\boldsymbol{X}$ has the intraclass correlation structure with positive correlation coefficient), then $F_{X^*}(x)$ intersects $\Phi(x)$ exactly once and this is from below. Therefore there exists some value $x_0 \in \mathbb{R}$ such that X^* dominates the standard Gaussian on the interval $(-\infty, x_0)$ but the standard Gaussian dominates X^* on (x_0, ∞).

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Multivariate normal random vectors with the intraclass correlation structure occur in random effects models in which the error in a measurement arises as a combination of a class-specific error and an individual-specific error. More precisely, $X_i = \mu_i + \sqrt{\rho}Y_0 + \sqrt{1-\rho}Y_i$ for $i = 1, \dots, k$, where $\rho \in (0, 1)$ and the $Y_0, \dots, Y_k$ are independent standard normal random variables. Our motivation for this work was an application to the Bayesian design of exploratory clinical trials in which k experimental treatments are compared to a single control (Whitehead et al., 2015). In that paper, one or more of the treatments is suitable to be developed further in a phase III trial if there is a sufficiently high probability that at least one treatment out-performs the control by a given threshold. Corollary 1 enables us to quantify the effect of increasing the threshold on that probability. This is then used to recommend an appropriate sample size for the trial.

The main results are stated in Section 2 and proved in Section 3. The proof is surprisingly long and technical, as well as being very sensitive to the assumptions. We are not aware of any simplifications to the argument, however, nor of other results in the literature that enable a comparison of this form.

2. Statement of results

In this section, we state our main theorem, which is then proved in Section 3. We also state and prove the corollary of this result that is used in Whitehead et al. (2015).

We begin with some notation. For $\rho \in (0, 1)$ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k) \in \mathbb{R}^k$, let $(X_1, \dots, X_k) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ be a multivariate Gaussian random vector with $\Sigma_{ij} = \rho + (1-\rho)\delta_{ij}$, where δ_{ij} is the Kronecker delta. Let $X^* = \max\{X_1, \dots, X_k\}$. For any random variable Y, we denote the distribution function of Y by F_Y and the density function of Y by f_Y. In the special case when $Y \sim N(0, 1)$, we set $\Phi = F_Y$ and $\phi = f_Y$.

Theorem 1. *For any $\rho \in (0, 1)$ and $\boldsymbol{\mu} \in (-\infty, 0)^k$, the distribution functions $F_{X^*}(x)$ and $\Phi(x)$ intersect exactly once. Furthermore, if $x_0 \in \mathbb{R}$ is the intersection point, then $f_{X^*}(x_0) > \phi(x_0)$.*

A direct consequence of this result is that

$$F_{X^*}(x) > \Phi(x) \quad \text{for all } x > x_0;$$
$$F_{X^*}(x) < \Phi(x) \quad \text{for all } x < x_0.$$

Equivalently, if $Z \sim N(0, 1)$, then the conditional distribution of $[X^*|X^* > x_0]$ is stochastically dominated by the conditional distribution of $[Z|Z > x_0]$ and the conditional distribution of $[X^*|X^* < x_0]$ stochastically dominates the conditional distribution of $[Z|Z < x_0]$.

Corollary 1. *For any $\rho \in (0, 1)$ and $\boldsymbol{\mu} \in \mathbb{R}^k$, if $\mathbb{P}(X_i < 0 \text{ for all } i = 1, \dots, k) \geq \kappa$ for some $\kappa \in (0, 1)$, then $\mathbb{P}(X_i < \Phi^{-1}(\zeta) - \Phi^{-1}(\kappa) \text{ for all } i = 1, \dots, k) > \zeta$ for all $\zeta \in (\kappa, 1)$.*

Proof of Corollary 1. We have

$$\begin{aligned}\Phi(\Phi^{-1}(\kappa)) &\leq \mathbb{P}(X_i < 0 \text{ for all } i = 1, \dots, k) \\ &= \mathbb{P}(X_i + \Phi^{-1}(\kappa) < \Phi^{-1}(\kappa) \text{ for all } i = 1, \dots, k) \\ &= F_{Y^*}(\Phi^{-1}(\kappa)),\end{aligned}$$

where $Y^* = \max\{X_1 + \Phi^{-1}(\kappa), \dots, X_k + \Phi^{-1}(\kappa)\}$. If $\zeta > \kappa$, then $\Phi^{-1}(\zeta) > \Phi^{-1}(\kappa)$ and so, applying Theorem 1 to $(X_1 + \Phi^{-1}(\kappa), \dots, X_k + \Phi^{-1}(\kappa))$, gives $F_{Y^*}(\Phi^{-1}(\zeta)) > \Phi(\Phi^{-1}(\zeta)) = \zeta$, as required. (Since $F_{X^*}(x) < \Phi(x)$ for all $x \in \mathbb{R}$ when $\mu_i \geq 0$ for any i, the existence of κ guarantees that $\mathbb{E}(X_i + \Phi^{-1}(\kappa)) < 0$ for all i.)

3. Proof of main result

In this section we provide a proof of Theorem 1. We begin by expressing $F_{X^*}(x)$ in terms of independent identically distributed (i.i.d.) Gaussian random variables. We then show that the distribution functions intersect at most once provided that a quantity expressed in terms of these i.i.d. variables can be shown to be strictly positive. This quantity is obtained as the solution to a first order linear differential equation with variable linear coefficient. Positivity follows by showing that the linear coefficient is negative. The coefficient is expressed in terms of standard univariate Gaussian density functions which then enables us to deduce positivity as a consequence of properties of the inverse Mill's ratio. Finally, we show that the distribution functions must intersect at least once, by considering their relative values in the limit as $x \to \pm\infty$.

For $\rho \in (0, 1)$, $\nu_0 \in \mathbb{R}$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_k) \in \mathbb{R}^k$, let $Y_0, \dots, Y_k$ be independent Gaussian random variables with $Y_0 \sim N(\nu_0, \rho)$ and $Y_i = N(\nu_i, 1-\rho)$ for $i = 1, \dots, k$. Let

$$G(\nu_0, \boldsymbol{\nu}) = \mathbb{P}(\max\{Y_1 - Y_0, \dots, Y_k - Y_0\} < 0).$$

Observe that $(Y_1 - Y_0, \dots, Y_k - Y_0) \sim N(\boldsymbol{\nu} - \nu_0, \boldsymbol{\Sigma})$ where $\boldsymbol{\Sigma}$ is as defined in Section 2 and hence $F_{X^*}(x) = G(x, \boldsymbol{\mu}) = G(0, \boldsymbol{\mu} - x)$.

As $G(\nu_0, \boldsymbol{\nu})$ is strictly increasing in $\nu_0 \in (-\infty, \infty)$ from 0 to 1, there exists a unique function $g : (0, 1) \times \mathbb{R}^k \to \mathbb{R}$ such that $G(g(\zeta, \boldsymbol{\nu}), \boldsymbol{\nu}) = \zeta$. So $F_{X^*}(x) = \Phi(x)$ for some $x \in \mathbb{R}$ if and only if $g(\zeta, \boldsymbol{\mu}) = \Phi^{-1}(\zeta)$ for some $\zeta \in (0, 1)$. In order to