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Complete asymptotic expansions for normal extremes

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ABSTRACT

Hall (1979) was the first to derive the rate of convergence for normal extremes. Many authors have followed up the work of Hall, but complete asymptotic expansions have not been known for normal extremes. Here, we derive such expansions for the first time. The expansions are *single* infinite sums of terms involving Bell polynomials and Stirling numbers. The usefulness of the expansions over the results in Hall is illustrated computationally.

1. Introduction

Let $\Phi(\cdot)$ denote the cumulative distribution function (cdf) of a standard normal random variable. It is well known that $\Phi(\cdot)$ belongs to the max domain of attraction of the Gumbel extreme value distribution, i.e.,

$$\Phi^n\left(a_n x + b_n\right) \to \exp\left\{-\exp(-x)\right\} \tag{1}$$

as $n \to \infty$, where

$$a_n = (2\log n)^{-1/2}, \qquad b_n = a_n^{-1} - \frac{a_n}{2}\left[\log\log n + \log(4\pi)\right] \tag{2}$$

for $n \geq 1$.

Many authors have been interested in convergence aspects of (1) because of the universality of the normal distribution. Hall (1979) was the first to investigate convergence aspects of (1). He established the convergence rate of (1). He chose a_n and b_n slightly differently to satisfy

$$a_n = b_n^{-1}, \qquad 2\pi b_n^2 \exp\left(b_n^2\right) = n^2 \tag{3}$$

for $n \geq 1$. Since Hall's seminal paper, many other authors have considered convergence aspects of (1). We mention Hall (1980), Nair (1981), Cohen (1982a,b), Rootzén (1983) and Gomes (1984).

To the best of our knowledge, complete asymptotic expansions for $\Phi^n\left(a_n x + b_n\right)$ have not been known to date. By a complete asymptotic expansion, we mean the following: suppose that

$$F_n(x) \to P(x)$$

uniformly as $n \to \infty$. Suppose that we have an asymptotic expansion of $P(x)$, so we can write

$$P(x) = \sum_{i=0}^{\infty} c_i e_i(x),$$

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where c_i is a constant and e_i is a function. Write

$$P_n(x) = \sum_{i=0}^{n} c_i e_i(x)$$

and define $\Delta_n(x) = F_n(x) - P_n(x)$. The complete asymptotic expansion is an expansion for $\Delta_n(x)$ of the form

$$\Delta_n(x) = \sum_{i=1}^{\infty} C_{i,n}\pi_{i,n}(x). \tag{4}$$

This expansion can be derived for any $F_n(x)$. Some recent examples of complete asymptotic expansions are those due to Hashorva (2009a,b, 2010), Debicki et al. (2014) and Hashorva et al. (2014).

An expansion such as (4) could have both practical and theoretical appeal. In a practical sense, it could lead to better approximations, see Section 3 for an illustration. Theoretically, such an expansion can be used to derive expansions for the corresponding probability density function (pdf), moments, cumulants, quantiles, etc.

The aim of this note is to derive complete asymptotic expansions for $\Phi^n(a_n x + b_n)$. The derived expansions are *single* infinite sums. The terms of the infinite sums involve Bell polynomials and the Stirling number of the first kind. In-built routines for Bell polynomials and the Stirling number of the first kind are available in most computer algebra packages. For example, see BellY and StirlingS1 in Mathematica. So, the expansions given will be accessible to most practitioners.

The complete asymptotic expansions for $\Phi^n(a_n x + b_n)$ are given in Section 2. The corresponding proof is given in Section 4. Computational issues relating to the expansions in Section 2 are discussed in Section 3.

Our results in Sections 2–4 can in principle be extended to any other cdf belonging to the max domain of attraction of an extreme value distribution. We have illustrated our results for the normal distribution because of its universality.

For $\mathbf{x} = (x_1, x_2, \ldots)$, we shall let $B_{rk}(\mathbf{x})$ denote the *partial exponential Bell polynomial* defined by

$$\left(\sum_{r=1}^{\infty} x_r t^r / r!\right)^k \Big/ k! = \sum_{r=k}^{\infty} B_{rk}(\mathbf{x}) t^r / r!. \tag{5}$$

This polynomial is tabled on p. 307 of Comtet (1974) for $r \leq 12$. We shall use the notation $(a)_k = a(a+1)\cdots(a+k-1)$ to denote the ascending factorial and the notation $S_i^{(j)}$ to denote the Stirling number of the first kind.

2. Main results

Our main results are Theorems 1 and 2. Theorem 1 gives a complete asymptotic expansion for

$$\Delta_n(x) = \Phi^n(a_n x + b_n) - \sum_{k=0}^{n} \frac{(-1)^k \exp(-kx)}{k!} \tag{6}$$

as $n \to \infty$ when a_n and b_n are given by (2). Theorem 2 gives a complete asymptotic expansion for

$$D_n(x) = \Phi^n(a_n x + b_n) - \sum_{k=0}^{n} \frac{(-1)^k \exp(-kx)}{k!} \tag{7}$$

as $n \to \infty$ when a_n and b_n are given by (3). The proof of Theorem 1 is given in Section 4. The proof of Theorem 2 is similar to that of Theorem 1.

Theorem 1. *With a_n and b_n given by* (2), *we have*

$$\begin{aligned}
\Delta_n(x) =& \sum_{r=1}^{\infty} \frac{(-2)^r B_{0r}(\mathbf{c})}{r!} (2\log n)^{-r} \Delta_{n,0,r}(x) \\
&+ \sum_{r=1}^{\infty}\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k (-2)^r B_{r0}(\mathbf{c})}{r!} \exp(-kx) n^{-k} (2\log n)^{-r} \Delta_{n,k,r}(x) \\
&+ n! \sum_{r=n}^{\infty}\sum_{k=1}^{n}\sum_{\ell=1}^{k} \frac{(-1)^k (-2)^r B_{r\ell}(\mathbf{c})}{(n-k)!\, r!\, (k-\ell)!} \exp(-kx) n^{-k} (2\log n)^{-r} \Delta_{n,k,r}(x) \\
&+ 2n! \sum_{r=1}^{n}\sum_{k=1}^{r}\sum_{\ell=1}^{k} \frac{(-1)^k (-2)^r B_{r\ell}(\mathbf{c})}{(n-k)!\, r!\, (k-\ell)!} \exp(-kx) n^{-k} (2\log n)^{-r} \Delta_{n,k,r}(x) \\
&+ \sum_{r=1}^{n} \frac{(-1)^r}{r!} \exp(-rx) \left[\Delta_{n,r,0}(x) - 1\right] \\
&+ \sum_{r=1}^{n}\sum_{\ell=1}^{r}\sum_{m=0}^{\ell-1} \frac{(-1)^{r+m+\ell} (1-r)_{r-\ell} S_\ell^{(m)}}{\ell!\,(r-\ell)!} \exp(-rx) n^{m-r} \Delta_{n,r,0}(x)
\end{aligned} \tag{8}$$