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Discrete strong unimodality of order statistics

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ABSTRACT

We show that discrete logconcavity of probability mass functions implies logconcavity of their distribution and survival functions. Applications are obtained and discrete strong unimodality of order statistics (OSs) is established. Illustrative examples are provided and finally discrete logconvexity of OSs is discussed.

1. Introduction

The notions of unimodality and strong unimodality exist for both discrete and continuous distributions, each one having its own interpretation. A cumulative distribution function F is said to be unimodal if there exists a value $x = a$ such that $F(x)$ is convex for $x < a$ and concave for $x > a$. Since the property of unimodality is not preserved under convolutions, Ibragimov (1956) introduced and characterized strongly unimodal distributions. F is said to be strongly unimodal if its convolution with any unimodal distribution is unimodal. Clearly, any strongly unimodal distribution is unimodal but the converse is not necessarily true. He proved that this is equivalent to F having a logconcave density f.

According to the above definition, the only unimodal discrete distributions are the degenerate ones. However, a discrete distribution with probability mass function $\{p_i\}$, on the lattice of integers, is admitted to be unimodal about a, if

$$\begin{cases} p_i \geq p_{i-1}, & \text{i} \leq a \\ p_i \geq p_{i+1}, & \text{i} \geq a. \end{cases}$$

$\{p_i\}$ is said to be strongly unimodal, if the convolution of $\{p_i\}$ with any discrete unimodal distribution $\{q_i\}$ is unimodal. As in the continuous case, Keilson and Gerber (1971) showed that a necessary and sufficient condition for $\{p_i\}$ to be strongly unimodal is that $\{p_i\}$ is a logconcave sequence, i.e.,

$$p_i^2 \geq p_{i-1}p_{i+1} \quad \forall i. \tag{1}$$

An excellent source for the details of these concepts is provided by Dharmadhikari and Joag-Dev (1988) (for convolutions of discrete logconcave random variables, see also Johnson and Goldschmidt, 2006).

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Because of the importance of ordered random variables (e.g., order statistics (OSs)) in many branches of statistical theory and applications, unimodality and strong unimodality of such random variables have been extensively studied in the literature in the continuous case (see, for example Alam, 1972; Huang and Ghosh, 1982; Basak and Basak, 2002; Aliev, 2003; Cramer et al., 2004; Cramer, 2004). The results of Chen et al. (2009) and Alimohammadi and Alamatsaz (2011) concerning generalized order statistics (for the definition see Kamps, 1995) unified most of these results. The recent work of Alimohammadi et al. (2014) almost exhausted these results in the continuous case. Here, we were motivated to investigate strong unimodality of OSs in the discrete case.

In this article, we obtain some basic and useful results and show that, similar to the continuous case, discrete logconcavity of a probability mass function (pmf) implies logconcavity of its cumulative distribution (cdf) and survival functions (sf). As an application, we shall show that discrete logconcave distributions are IFR (increasing failure rate) and DRHR (decreasing reverse hazard rate) and discrete logconvex ones are DFR (decreasing failure rate) while they cannot be IRHR (increasing reverse hazard rate). Then, using these fundamental results, we shall establish discrete strong unimodality of OSs by means of the notion of total positivity and a lemma due to Misra and van der Meulen (2003). Some illustrative examples are also provided in this regard. Finally, we obtain some results about discrete logconvexity of OSs.

Throughout the paper, increasing and decreasing mean non-decreasing and non-increasing, respectively. Further, ratios are supposed to be well defined whenever they are used.

2. Some preliminaries results

Let X be a discrete random variable with support $S_X = \{L, \ldots, U\} \subseteq \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, pmf $p_i = P(X = i)$, cdf $F_i = P(X \le i)$, and sf $\bar{F}_i = P(X > i)$, $i \in S_X$.

In this section, we review some useful related results. We first need the following comprehensive fundamental lemma which is an interesting result on its own. Some parts of the lemma are known in the literature but they are recalled and proved for completeness. First, we note that it is not difficult to see that if $\{p_i\}$ is logconvex, i.e., $p_i^2 \le p_{i-1}p_{i+1}$, $\forall i$, then $U = \infty$. Furthermore, in Theorem 2.3, we shall show that if $\{p_i\}$ is logconvex, then $p_i/\bar{F}_{i-1}$ is decreasing which in turn implies that $\{p_i\}$ is decreasing (and thus unimodal). Therefore, L must be finite because of the summability of $\{p_i\}$. Thus, in the rest of the article, we assume that the support of a logconvex pmf $\{p_i\}$ is $S_X = \{L, \ldots, \infty\}$ with $L > -\infty$.

Lemma 2.1. *Let $\{p_i\}$ be a logconcave (logconvex) pmf and $i, j \in S_X$. Then, we have:*

(i) $p_ip_j \ge (\le) p_{i-1}p_{j+1}$, *for* $j \ge i$;
(ii) $F_ip_j \ge (\le) F_{i-1}p_{j+1}$, *for* $j \ge i - 1$;
(iii) $p_i\bar{F}_{j-1} \ge (\le) p_{i-1}\bar{F}_j$, *for* $j \ge i - 1$.

For logconvexity, the extra condition of $i - 1 \in S_X$ is needed on the boundary of the support.

Proof. Assume that $\{p_i\}$ is logconcave. Then, we have:

(i) The assertion is obvious by noticing that logconcavity of the sequence $\{p_i\}$ implies that $\{p_i/p_{i-1}\}$ is decreasing.
(ii) From (i), for $j \ge i$ we have

$$\sum_{k=-\infty}^{i} p_kp_j \ge \sum_{k=-\infty}^{i} p_{k-1}p_{j+1} \Longrightarrow F_ip_j \ge F_{i-1}p_{j+1}, \quad j \ge i. \tag{2}$$

For $i = j$, (2) yields that

$$\frac{F_{j+1}}{p_{j+1}} = 1 + \frac{F_j}{p_{j+1}} \ge \frac{F_{j-1}}{p_j} + 1 = \frac{F_j}{p_j}.$$

So, (2) is also valid for $j \ge i - 1$. Thus, (ii) is true.
(iii) Again, from (i), for $j \ge i$ we have

$$\sum_{k=j}^{\infty} p_ip_k \ge \sum_{k=j}^{\infty} p_{i-1}p_{k+1} \Longrightarrow p_i\bar{F}_{j-1} \ge p_{i-1}\bar{F}_j, \quad j \ge i. \tag{3}$$

Now, letting $j = i$ in (3) we obtain

$$\frac{\bar{F}_{i-2}}{p_{i-1}} = 1 + \frac{\bar{F}_{i-1}}{p_{i-1}} \ge \frac{\bar{F}_i}{p_i} + 1 = \frac{\bar{F}_{i-1}}{p_i}.$$

Thus, (3) follows for every $j \ge i - 1$. The results for logconvexity are proved similarly. □

To establish our main results, we first present a useful theorem concerning the inherence of discrete logconcavity from a pmf to its cdf and sf.