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A ratio goodness-of-fit test for the Laplace distribution

Elizabeth González-Estrada *, José A. Villaseñor *

Colegio de Postgraduados, Km. 36.5 Carr. México-Texcoco, Montecillo, 56230, Mexico

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ABSTRACT

A test based on the ratio of the sample mean absolute deviation and the sample standard deviation is proposed for testing the Laplace distribution hypothesis. The asymptotic null distribution for this test statistic is found to be normal. The use of Anderson–Darling test based on a data transformation is also discussed.

1. Introduction

A random variable X has the Laplace distribution, also called double exponential distribution, with location and scale parameters $-\infty < \theta < \infty$ and $\beta > 0$, denoted by $X \sim L(\theta, \beta)$, if its cumulative distribution function (cdf) is given by

$$\begin{aligned} F_X(x) &= \frac{1}{2} \exp\left\{-\frac{\theta - x}{\beta}\right\}, \quad x \le \theta, \\ &= 1 - \frac{1}{2} \exp\left\{-\frac{x - \theta}{\beta}\right\}, \quad x \ge \theta. \end{aligned}$$

The mean and variance of X are θ and $\sigma_X^2 = 2\beta^2$. This distribution is closely related to the exponential distribution with cdf $F(z) = 1 - \exp\{-z/\beta\}$, where $z > 0$ and $\beta > 0$, which is denoted as $Exp(\beta)$. In fact, if $X \sim L(\theta, \beta)$ then the random variable

$$Y^{(\theta)} = |X - \theta| \sim Exp(\beta). \tag{1}$$

Some applications of the Laplace family of distributions are found in the areas of economics, finance, health sciences, hydrology, etc. (Kotz et al., 2001; Puig and Stephens, 2000), where it is used for modeling symmetric datasets. In this paper we consider the problem of testing the null hypothesis H_0 : a random sample $X_1, \ldots, X_n$ comes from a $L(\theta, \beta)$ distribution with unknown parameters. This topic has also been addressed by Puig and Stephens (2000), Meintanis (2004), Choi and Kim (2006), Best et al. (2008), Gel (2010) and Lafaye de Micheaux and Tran (2016), among others. Current references on goodness-of-fit are Torabi et al. (2016) and Roberts (2015).

For testing H_0 we propose a test based on the ratio of two estimators for the scale parameter β: the sample mean absolute deviation and $1/\sqrt{2}$ times the sample standard deviation. A similar approach has been used before by Geary (1936) and Gel

* Corresponding authors.
E-mail address: egonzalez@colpos.mx (E. González-Estrada).

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et al. (2007) for testing normality. On the other side, as a second test for H_0, using property in (1) we also propose to transform $X_1, \dots, X_n$ to approximately exponential random variables and then use Anderson–Darling test for testing exponentiality.

Puig and Stephens (2000) considered the Anderson–Darling, Watson and Cramér–von Mises tests for testing H_0, which compare the empirical distribution function (EDF) to the Laplace cumulative distribution function. Their power studies indicate that, among the EDF tests, Watson test is in general the best test against symmetric distributions and that Anderson–Darling test performs poorly against this kind of alternatives. The simulation results presented in Section 3 show that Anderson–Darling test performs better than Watson test when it is based on transformed observations instead of the original observations. These results also indicate that the ratio test is powerful against symmetric alternative distributions.

This manuscript is organized as follows. In Section 2 the proposed tests are presented and the asymptotic null distribution of the ratio test is obtained. The results of a Monte Carlo simulation study conducted in order to assess the power properties of the tests are presented in Section 3. Some conclusions are provided in Section 4.

2. New tests for the Laplace distribution

Let $X_1, \dots, X_n$ be a random sample of size n from a continuous population with cdf F. Next we present two tests for the composite null hypothesis:

$$H_0 : X \sim L(\theta, \beta), \tag{2}$$

where $-\infty < \theta < \infty$ and $\beta > 0$ are unknown.

2.1. A test based on the ratio of two estimators for β

Notice that an estimator of the scale parameter β is the sample mean absolute deviation (MAD) about the sample mean $\bar{X}_n$, defined as

$$\check{\beta}_n = \sum_{i=1}^{n} |X_i - \bar{X}_n|/n. \tag{3}$$

On the other hand, a moments estimator of β is $\tilde{\beta}_n = \sqrt{S_n^2/2}$, where S_n^2 is the sample variance.

If H_0 in (2) holds, then the following statistic:

$$R_n = \tilde{\beta}_n/\check{\beta}_n \tag{4}$$

is expected to take on values close to one. Therefore, a test for the Laplace distribution based on R_n rejects H_0 at a significance level $\alpha \in (0, 1)$ if $R_n < c_{\alpha/2}$ or $R_n > c_{1-\alpha/2}$, where these critical values satisfy the following equation:

$$1 - \alpha = P(c_{\alpha/2} < R_n < c_{1-\alpha/2} | H_0), \tag{5}$$

since R_n is a location-scale invariant statistic. Critical values for this test can be obtained approximately from the asymptotic null distribution of R_n; however, for small sample sizes these values can be computed by Monte Carlo simulation for each sample size n.

Next theorem provides the asymptotic null distribution of R_n.

Theorem 1. *Under H_0, $\sqrt{4n}(R_n - 1) \xrightarrow{d} \eta(0, 1)$, as $n \to \infty$.*

For the proof of this theorem, see the Appendix A section. □

Corollary 1. *Under H_0, $\sqrt{n}(\check{\beta}_n - \beta)/\beta \xrightarrow{d} N(0, 1)$, as $n \to \infty$.* □

Remark 1. From expression (A.9) in the Appendix A, a well-known result (Kotz et al., 2001) follows: $\sqrt{n}(S_n/\sqrt{2} - \beta) \xrightarrow{d} N(0, 5\beta^2/4)$. □

From Theorem 1, a test based on $R_n^* = \sqrt{4n}(R_n - 1)$ rejects H_0 at a test size $\alpha \in (0, 1)$ when the sample size is large if $|R_n^*| > z_{1-\alpha/2}$, where $z_{1-\alpha/2}$ is the $100(1 - \alpha/2)$% quantile of the standard normal distribution.

Remark 2. An additional test for the Laplace distribution can be based on the ratio $R'_n = \tilde{\beta}_n/\hat{\beta}_n$, where $\hat{\beta}_n = \sum_{i=1}^{n} |X_i - \hat{\theta}_n|/n$ is the maximum likelihood estimator of β and $\hat{\theta}_n$ is the sample median. □

2.2. Anderson–Darling test based on a data transformation

Under H_0, $Y^{(\theta)} = |X - \theta|$ is distributed as $Exp(\beta)$. Hence, if the unknown value of the location parameter θ is replaced by the sample mean $\bar{X}_n$, then the transformed observations $Y_i = |X_i - \bar{X}_n|,\ i = 1, \dots, n$, are asymptotically independent